Part in class 984

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from his sincere friend and admirer

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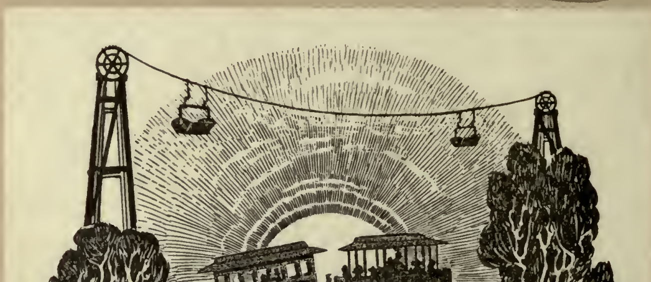

# ADVANCED

# ALGEBRA

BY

WM. T. WELCKER

*Graduate of West Point,*

PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF CALIFORNIA,

AUTHOR OF

"WELCKER'S MILITARY LESSONS," "WELCKER'S PRIMARY ARITHMETIC" (UNPUBLISHED), "WELCKER'S PRACTICAL ARITHMETIC" (UNPUBLISHED), ETC.



SAN FRANCISCO:

W. M. HINTON & CO., BOOK AND JOB PRINTERS

536 CLAY STREET.

# PREFACE.

This small volume contains what remains of the course in Algebra, after matriculation, to the students in the Colleges of Civil Engineering, Mines, and Mechanic Arts in the University of California.

It is intended as a continuation of the excellent work on algebra by Mr. John B. Clarke, of the Mathematical Department of the University; and it is thought it will, in connection with Clarke's Algebra, or with any work of similar scope, furnish a good and sufficient preparation for those who intend to pursue the higher mathematics.

The constant aim and endeavor throughout has been so to present the various topics discussed as to render them easy of comprehension by the undergraduate student.

WM. T. WELCKER.

Berkeley, California,
July, 1880.

# CONTENTS.

# PRINCIPLES OF ALGEBRA.

## CHAPTER I.

### Summation of Series.

**Art. 1.** In addition to the treatment of this subject in Clark's Algebra, a little more will be here added. It was there shown that when a series was so constituted that each term might be derived from the expression $\frac{q}{k(k+p)}$ by giving a constant value to $p$ and suitable values to $q$ and $k$, the sum of $n$ terms of the series could be found by forming two auxiliary series from the expressions $\frac{q}{k}$ and $\frac{q}{k+p}$ respectively, subtracting $n$ terms of the second from the corresponding $n$ terms of the first, and then dividing the difference by $p$.

**Art. 2.** *If a series of fractions have the form* $\frac{q}{k(k+p)(k+2p)}$ *its sum is equal to the difference between a series whose terms have the form* $\frac{q}{k(k+p)}$ *and another whose terms have the form* $\frac{q}{(k+p)(k+2p)}$ *divided by* $2p$.

For

$$\frac{q}{k(k+p)}-\frac{q}{(k+p)(k+2p)}=\frac{q(k+p)(k+2p)-qk(k+p)}{k(k+p)^2(k+2p)}=$$

$$\frac{q[k^2+3pk+2p^2-k^2-pk]}{k(k+p)^2)(k+2p)}=\frac{q\,.\,2p}{k(k+p)(k+2p)},$$ and this divided by $2p$ gives the form proposed. If any term of the pro-

posed can be found in this way, the sum of $n$ terms of that series may be had by taking the difference between the sums of the $n$ corresponding terms of the two auxiliary series and dividing it by $2p$.

Ex. 1. What is the sum of the series

$$\frac{3}{5.8.11}+\frac{9}{8.11.14}+\frac{15}{11.14.17}+\frac{21}{14.17.20}+\text{etc ?}$$

Comparing with the formula above, we see that $q=3$, 9, 15, etc.; $p=3$; and $k=5$, 8, 11, 14, etc. Therefore the auxiliary series are:

$$\text{From } \frac{q}{k(k+p)}: \quad \frac{3}{5.8}+\frac{9}{8.11}+\frac{15}{11.14}+\ \cdots\cdots+\frac{3(2n-3)}{(3n-1)(3n+2)}+\frac{3(2n-1)}{(3n+2)(3n+5)}$$

$$\text{From } \frac{q}{(k+p)(k+2p)}: \quad \frac{3}{8.11}+\frac{9}{11.14}+\ \cdots\cdots+\frac{3(2n-3)}{(3n+2)(3n+5)}+\frac{3(2n-1)}{(3n+5)(3n+8)}$$

and the sum:

$$S=\frac{1}{2.3}\left[\frac{3}{5.8}-\frac{3(2n-1)}{(3n+5)(3n+8)}\right]+\frac{1}{8.11}+\frac{1}{11.14}+\frac{1}{14.17}+\ \cdots\cdots \text{ to infinity.}$$

Now the sum of this last series to infinity $=\frac{1}{24}$. Hence the sum of $n$ terms of the proposed series is:

$$\frac{1}{2}\left[\frac{1}{40}-\frac{2n-1}{(3n+5)(3n+8)}\right]+\frac{1}{24},$$

and when $n=\infty$,

$$S=\frac{1}{2}\left[\frac{1}{40}-\frac{\frac{2}{n}-\frac{1}{n^2}}{\left(3+\frac{5}{n}\right)\left(3+\frac{8}{n}\right)}\right]+\frac{1}{24}=\frac{1}{80}+\frac{1}{24}=\frac{13}{240}.$$

The sum of $n$ terms of Example 1 =

$$\frac{1}{2}\left[\frac{1}{40}-\frac{2n-1}{(3n+5)(3n+8)}\right]+\frac{1}{8.11}+\frac{1}{11.14}+\ldots \text{ to } n-1 \text{ terms}$$

Ex. 2. What is the sum of an infinite number of terms of the series $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+$ etc?

Ans. $1\frac{1}{4}$

Ex. 3. What is the sum of the series $\frac{1}{1.3.5}+\frac{4}{3.5.7}+\frac{7}{5.7.9}+\frac{10}{7.9.11}+$ etc., to infinity? Ans. $\frac{5}{24}$.

**Art. 3.** If a series of fractions have the terms of the form $\frac{q}{k(k+p)(k+2p)(k+3p)}$, its sum will be equal to the difference between the sums of two series whose terms have respectively the forms

$$\frac{q}{k(k+p)(k+2p)} \text{ and } \frac{q}{(k+p)(k+2p)(k+3p)}$$

divided by $3p$. In these formulas $p$ is constant, and $q$ and $k$ have suitable values assigned to them. The proof of the above is easily seen by performing the subtraction:

$$\frac{q}{k(k+p)(k+2p)}-\frac{q}{(k+p)(k+2p)(k+3p)}=$$
$$\frac{(k+p)(k+2p)(qk+3pq)-(k+p)(k+2p)qk}{k(k+p)^2(k+2p)^2(k+3p)}=$$
$$\frac{3pq}{k(k+p)(k+2p)(k+3p)}.$$

Ex. 4. What is the sum of $n$ terms and of an infinite number of terms of the series $\frac{6^2}{1.2.3.4}+\frac{7^2}{2.3.4.5}+\frac{8_2}{3.4.5.6}+$ etc?

Here $q = 6^2,\ 7^2,\ 8^2,\ 9^2$, etc.; $p = 1$; $k = 1, 2, 3, 4, 5$, etc. Hence the auxiliary series are:

$$\frac{6^2}{1.2.3} + \frac{7^2}{2.3.4} + \frac{8^2}{3.4.5} + \quad .\ .\ .\ .\ .\quad \frac{(n+5)^2}{n(n+1)(n+2)}$$

$$\frac{6^2}{2.3.4} + \frac{7^2}{3.4.5} + \quad .\ .\ .\ .\ .\quad \frac{(n+4)^2}{n(n+1)(n+2)} + \frac{(n+5)^2}{(n+1)(n+2)(n+3)}$$

and $S = \frac{1}{3}\left[\frac{6^2}{1.2.3} - \frac{(n+5)^2}{(n+1)(n+2)(n+3)}\right] + \frac{1}{3}\left(\frac{13}{2.3.4} + \frac{15}{3.4.5} + \frac{17}{4.5.6}\text{ to } n \text{ terms.}\right)$

The sum of this last series to infinity is, by preceding methods $= \frac{17}{12}$;

$$\therefore\ S = \frac{1}{3}\left[6 - \frac{\left(\frac{1}{n^2} + \frac{5}{n^3}\right)(n+5)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)}\cdot\right]$$

$$+ \frac{1}{3}\left(\frac{13}{2.3.4} + \frac{15}{3.4.5} + \frac{17}{4.5.6} + \text{etc., to } n \text{ terms.}\right)$$

When $n = \infty$, $S = \frac{1}{3}\left(6 + \frac{17}{12}\right) = \frac{89}{36}$.

Ex. 5. What is the sum of the series

$\frac{1}{1.3.5.7} + \frac{2}{3.5.7.9} + \frac{3}{5.7.9.11}$, to infinity? Ans. $\frac{1}{72}$.

**Art. 4.** A consideration of the laws of the preceding series and the mode of their summation will show that the sum of any series of fractions of the form

$$\frac{q}{k(k+p)(k+2p)\ .\ .\ .\ .\ (k+mp)}$$

is equal to $\frac{1}{mp}$ of the difference between two series of fractions of the forms, respectively,

$$\frac{q}{k(k+p)\ .\ .\ [k+(m-1)p]} \text{ and } \frac{q}{(k+p)(k+2p)\ .\ .\ .\ [k+mp]}$$

**Art. 5.** If a series of fractions has the terms as follow:

$$\frac{c}{k}+\frac{c(c+p)}{k(k+p)}+\frac{c(c+p)(c+2p)}{k(k+p)(k+2p)}+\ .\ .\ +\frac{c(c+p)\ .\ .\ .\ (c+mp)}{k(k+p)\ .\ .\ .\ (k+mp)}$$

(in which the last term is a *general* term), its sum will be equal to the difference of two series of fractions, having respectively the forms:

$$\frac{c(c+p)\ .\ .\ .\ (c+mp)}{k(k+p)\ .\ .\ .\ [k+(m-1)p]} \text{ and } \frac{c(c+p)\ .\ .\ .\ [c+(m+1)p]}{k(k+p)\ .\ .\ .\ (k+mp)}$$

divided by $k-c-p$, as will be seen by performing the indicated subtraction.

Ex. 1. Find the sum of $m$ terms of the series $\frac{2}{3}+\frac{2.4}{3.5}+\frac{2.4.6}{3.5.7}+$ etc.

Here $p=2$, $c=2$ and $k=3$, and the auxiliary series are:

$$2+\frac{2.4}{3}+\frac{2.4.6}{3.5}+\ .\ .\ .\ \frac{2.4.6\ .\ .\ .\ .\ [2+2m-2]}{3.5.7\ .\ .\ .\ [3+2m-4]} \text{ and}$$

$$\frac{2.4}{3}+\frac{2.4.6}{3.5}+\ .\ .\ .\ \frac{2.4.6\ .\ .\ .\ .\ 2m+2}{3.5.7\ .\ .\ .\ .\ (3+2m-2)};$$

$$S=\frac{1}{-1}\left(2-\frac{2.4.6\ .\ .\ .\ .\ (2m+2)}{3.5.7\ .\ .\ .\ .\ (2m+1)}\right)=$$

$$\frac{2.4.6\ .\ .\ .\ .\ (2m+2)}{3.5.6\ .\ .\ .\ .\ (2m+1)}-2.$$

Ex. 2. Find sum of $m$ terms of $\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+$ etc.

Ans. $\frac{1.3.5.7\ .\ .\ .\ (2m+1)}{2.4.6\ .\ .\ .\ 2m}-1.$

**Art. 6.** Some special modes of summation:

METHOD OF DE MOIVRE.

*Assume a series whose terms converge to 0 and involve the powers of an indeterminate $x$; place the series equal to $S$ and multiply both members of this equation by a suitable binomial, trinomial, etc., which involves the powers of $x$ with constant coefficients; then assume $x$ so that the binomial, trinomial, etc., may be equal to zero and transpose some of the first terms;*

*their sum will be found equal to the sum of the remaining terms.*

Ex. 1. Let it be required to find the sum of $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$ etc., to infinity. Place $S=1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+$ etc., and multiply both members by $x-1$. We get

$$S\,(x-1)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\text{ etc.}$$
$$-1-\frac{x}{2}-\frac{x^2}{3}-\frac{x^3}{4}-\frac{x^4}{5}-\text{ etc.}$$

Whence by addition $S\,(x-1)=-1+\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}$ + etc.

Now suppose that $x=1$ and we get $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$ + etc. = 1.

Here the first term, — 1, is transposed and is equal to the sum of the proposed series to infinity.

Ex. 2. If we had multiplied by the binomial $x^2-1$ we would have found the sum of $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\frac{1}{4.6}+$etc. and of $\frac{1}{1.3}-\frac{1}{2.4}+\frac{1}{3.5}-$ etc., to infinity.

Ex. 3. Let the multiplier be the binomial $2x-1$.

$$S=\quad 1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{4}+\text{ etc. to infinity}$$
$$2x-1$$

$$S\,(2x-1)=\quad 2x+\frac{2x^2}{2}+\frac{2x^3}{3}+\frac{2x^4}{4}+\text{ etc.}$$
$$-1-\frac{x}{2}-\frac{x^2}{3}-\frac{x^3}{4}-\frac{x^4}{5}+\text{ etc.; adding we obtain}$$

$S\,(2x-1)=-1+\frac{3x}{1.2}+\frac{4x^2}{2.3}+\frac{5x^3}{3.4}+\frac{6x^4}{4.5}+$ etc. Now make $2x-1=0$, whence $x=\frac{1}{2}$ and we get $-1+\frac{3}{1.2.2}+\frac{4}{2.3.2^2}+\frac{5}{3.4.2^3}+\frac{6}{4.5.2^4}+$ etc. $=0$, and the sum of the series $\frac{3}{1.2.2}+\frac{4}{2.3.2^2}+\frac{5}{3.4.2^3}+\frac{6}{4.5.2^4}+$ etc. to infinity $=1$.

**Art. 7.** It will be observed that when the multiplier is a binomial the resulting series, before the value of $x$ is assigned, will consist of fractions having *two* factors in the denominators; if the multiplier is a trinomial there will be *three* factors, etc.

Ex. 4. The sum of the series $\frac{5}{1.2.3.2^2}+\frac{6}{2.3.4.2^3}+$ $+\frac{7}{3.4.5.2^4}+$ etc., to infinity $=\frac{1}{4}$.

To prove this let the multiplier be $(2x—1)\ (x—1)=2x^2$ $—3x+1$: then

$$S\,(2x^2—3x+1)=2x^2+\frac{2x^3}{2}+\frac{2x^4}{3}+\frac{2x^5}{4}+\text{ etc.}$$

$$—3x—\frac{3x^2}{2}—\frac{3x^3}{3}—\frac{3x^4}{4}—\frac{3x^5}{5}—\text{ etc.}$$

$$+1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\frac{x^4}{5}+\frac{x^5}{6}+\text{ etc., which by addition is}$$

$$S\ (2x^2—3x+1)=1—\frac{5x}{1.2}+\frac{5x^2}{1.2.3}+\frac{6x^3}{2.3.4}+\frac{7x^4}{3.4.5}+\frac{8x^5}{4.5.6}+$$

etc. Now if we make the factor $2x—1=0$, we get $x=\frac{1}{2}$ and $\frac{5}{1.2.3.2^2}+\frac{6}{2.3.4.2^3}+\frac{7}{3.4.5.2^4}+$ etc. $=\frac{5}{1.2.2}—1=\frac{5}{4}$ $—1=\frac{1}{4}$.

Had we made the second factor equal 0 or $x=1$ we should have found the sum of the series $\frac{5}{1.2.3}+\frac{6}{2.3.4}+$ $\frac{7}{3.4.5}+$ etc. $=\frac{5}{2}—1=\frac{3}{2}$.

Ex. 5. Suppose $\frac{1}{m}+\frac{x}{m+r}+\frac{x^2}{m+2r}+\frac{x^3}{m+3r}+$ etc. $=S$ and that we use the multiplier $ax—b$, we will get

$$S\,(ax—b)=\begin{cases}\frac{ax}{m}+\frac{ax^2}{m+r}+\frac{ax^3}{m+2r}+\text{ etc.}\\ —\frac{b}{m}—\frac{bx}{m+r}—\frac{bx^2}{m+2r}—\frac{bx^3}{m+3r}—\text{ etc.}\end{cases}$$

And by addition we find $S(ax-b) = -\frac{b}{m} + \frac{(m+r)a - mb}{m(m+r)}x + \frac{(m+2r)a - (m+r)b}{(m+r)(m+2r)}x^2 +$ etc. Now make $ax - b = 0$, but retaining for the present $x$ in the second member we have after transposing $-\frac{b}{m}$ the series $\frac{(m+r)a - mb}{m(m+r)}x + \frac{(m+2r)a - (m+r)b}{(m+r)(m+2r)}x^2 + \frac{(m+3r)a - (m+2r)b}{(m+2r)(m+3r)}x^3 +$ etc. $= \frac{b}{m}$. Now if any particular numerical series be proposed and it can be shown to coincide with the above when suitable values are substituted for the various letters we will be able to find its sum from the formula.

Suppose it were required to sum the series $\frac{2}{1.3}\cdot\frac{1}{3} + \frac{3}{3.5}\cdot\frac{1}{3^2} + \frac{4}{5.7}\cdot\frac{1}{3^3} +$ etc. By inspection we find that $x = \frac{1}{3}$ and since $ax - b = 0$, $a = 3b$; $m = 1$, $r = 2$ and from the first numerator $(m+r)a - mb = 2 \therefore 3a - b = 2$; or $3a - \frac{1}{3}a = 2 \therefore \frac{8}{3}a = 2$, and $a = \frac{3}{4}$ and $b = \frac{1}{4}$. These several values being substituted in the formula, term after term, build up the proposed series, the sum of which therefore $= \frac{b}{m} = \frac{\frac{1}{4}}{1} = \frac{1}{4}$.

The foregoing gives the sum of the series to infinity, but if it be required to find the sum of a finite number, $n$, of terms we may proceed as follows:

Take the series to $n+1$ terms, and multiply by $ax-b$ as before,

$$\frac{1}{m} + \frac{x}{m+r} + \frac{x^2}{m+2r} + \cdots + \frac{x^{n-1}}{m+(n-1)r} + \frac{x^n}{m+nr} = S'$$

$$\frac{ax-b}{\phantom{x}}$$

$$\frac{ax}{m} + \frac{ax^2}{m+r} + \cdots\cdots\cdots + \frac{ax^{n-1}}{m+(n-1)r} + \frac{ax^n}{m+nr}$$

$$-\frac{b}{m} - \frac{bx}{m+r} - \frac{bx^2}{m+2r} - \cdots\cdots - \frac{bx^n}{m+nr} = (ax-b)S'$$

whence by adding and transposing we get:

$$\frac{(m+r)a-mb}{m(m+r)}x+\frac{(m+2r)a-(m+r)b}{(m+r)(m+2r)}x^2+\text{etc., to } n\text{th term}=$$

$S'(ax-b)+\frac{b}{m}-\frac{ax^{n-1-1}}{m+nr}$. Then, using the values of the letters belonging to the numerical series last considered, to wit: $m=1$, $r=2$, $x=\frac{1}{3}$, $a=\frac{3}{4}$, $b=\frac{1}{4}$, and we have

$$S=\text{sum of } n \text{ terms of } \frac{2}{1.3}\cdot\frac{1}{3}+\frac{3}{3.5}\cdot\frac{1}{3^2}+\frac{4}{5.7}\cdot\frac{1}{3^3}+\ \cdot\ \cdot\ \cdot=$$

$$\frac{1}{4}-\frac{1}{4.3^n(1+2n)}.$$

Ex. 6. Let it be required to find the sum of the infinite series $\frac{19}{1.2.3}\cdot\frac{1}{4}+\frac{28}{2.3.4}\cdot\frac{1}{8}+\frac{39}{3.4.5}\cdot\frac{1}{16}+\frac{52}{4.5.6}\cdot\frac{1}{32}+\text{etc.}$ Here $\frac{1}{2}$, the square and higher powers of which are present, is represented by $x$ in the series $1+\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\text{etc.}$ Moreover, there are *three* factors in the denominators of the coefficients. Let us, then, multiply by the trinomial $ax^2-bx+c$, and we get

$$S(ax^2-bx+c)=\left\{\begin{array}{l} +c+\frac{cx}{2}+\frac{cx^2}{3}+\frac{cx^3}{4}+\frac{cx^4}{5}+\text{etc.} \\ \qquad -bx-\frac{bx^2}{2}-\frac{bx^3}{3}-\frac{bx^4}{4}-\text{etc.} \\ \qquad\qquad ax^2+\frac{ax^3}{2}+\frac{ax^4}{3}+\text{etc.} \end{array}\right.$$

Whence adding: $c+\frac{c-2b}{1.2}x+\frac{6a-3b+2c}{1.2.3}x^2+$

$\frac{12a-8b-6c}{2.3.4}x^3+\frac{20a-15b+12c}{3.4.5}x^4+\text{etc., to infinity}=$ $S(ax^2-bx+c)$.

Now making $ax^2-bx+c=0$, and in substituting that value of $x=\frac{1}{2}$, previously quoted, we get $\frac{a}{4}-\frac{b}{2}+c=0$; also, from the first and second numerators of the proposed

series, compared with those over the same denominators in the formulas, we get two more equations: $6a-3b+2c=19$ and $12a-8b+6c=28$. From these we find $a=6$, $b=7$, $c=2$, and these placed in the formula, build up the proposed series. After transposing the first two terms, 2 and —3, we find: $\frac{19}{1.2.3}\cdot\frac{1}{4}+\frac{28}{2.3.4}\cdot\frac{1}{8}+\frac{29}{3.4.5}\cdot\frac{1}{16}+$ etc., to infinity $=3-2=1$.

We may determine the sum of $n$ terms of this series in the manner of the last example. It is:

$$1-\frac{4+n}{(n+1)(n+2)2^{n+1}}.$$

To determine the multiplier to be used for any particular case, assume the series up to the term $\frac{x^{n+1}}{n+2}$.

Ex. 7. Find the sum of $x+2x^2+3x^3+4x^4+$ etc., to $\infty$.

Multiply the proposed series by $x^2-2x+1$.

$$S=\frac{x}{1-2x+x^2}$$

Ex. 8. Find the sum of $x+4x^2+9x^3+16x^4+$etc., to $\infty$.

Use $1-3x+3x^2-x^3$ as a multiplier of the series proposed. Ans. $\frac{x(1+)x}{(1-x)^3}$.

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## CHAPTER II.

### Elimination.

**Art. 8.** In order that an equation may be solved, it must contain only one unknown quantity. If it is *identical* it may have an infinite number of solutions, but if it is a *common* equation it will have only a limited number of roots. If then we have a single equation containing more than one unknown quantity, and the equation is not identical, but is indeterminate, we must attribute values to all the unknown quantities save one, before solution. If we have two or more simultaneous equations we must eliminate so as to obtain a single equation with one unknown quantity.

The methods which are used when the equations are of the first degree will, upon application to those of the higher degrees, be found to fail to give results practically useful. The *method of the greatest common* divisor is that usually employed with higher equations, and to its discussion we will proceed after establishing a principle in the nature of equations which is

Art. 9. *If $a$ is a root of an equation whose second member is zero, the first member will be exactly divisible by the binomial $x - a$.*

Let the equation be $x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} + \dots + Tx + U = 0$, and let $a$ be a root of the equation; then $x = a$ or, $x - a = 0$.

Suppose the first member to be divided by $x - a$ and that we continue getting terms of the quotient until the remainder is without an $x$, *or is independent of $x$.* Let this remainder $= R$, and the quotient (which may consist of one or of several terms) $= Q$, then $(x - a)\,Q + R = x^m + Px^{m-1} + Qx^{m-2} + \dots + Tx + U = 0$. But since the second member $= 0$ the first must $= 0$, and since $a$ is by supposition a root; $x - a = 0$, and this leaves $R = 0$. The remainder being $= 0$, the division was exact, which proves the theorem.

*The converse* is also true. *If the first member of an equation whose second is zero, is divisible exactly by the binomial $x - a$, then $a$ is a root of the equation.*

Since the division is exact there will be no remainder, and $(x - a)\,Q = x^m + Px^{m-1} + Qx^{m-2} \dots + Tx + U$. Now whatever value of $x$ makes the second member of this equation $= 0$ is a root of the proposed equation; but $x = a$ effects this, by reducing the first member to zero, and hence $a$ is a root of the proposed equation.

These properties belong of right to the general Theory of Equations, to the discussion of which we will soon pass, but being necessary to the understanding of the principles of elimination now to be examined have been introduced here.

**Art. 10.** When two or more equations are simultaneous they have values for the unknown quantities entering them which are the same in all the equations. These are of course common to them all; and there may be values in the different equations which are not common. The common roots make the equations *compatible* and are known as *compatible roots*.

If thus $b$ were a root common to two equations in $x$ and $y$, and if $b$ were substituted for $y$ in the first members of the two equations, they would become polynomials in $x$ only; moreover, since $b$ is compatible with some value of $x$ in both the first members, if we call that value of $x = a$ then those first members will have, from Art. 9, a common divisor, $x - a$. Having substituted in the two first members the known value of $y$, let it be supposed that the process for obtaining the H. C. D. were applied to them; it would terminate of course in a remainder $= 0$.

If, then, without substituting the value of $y$, and even without knowing it, we apply the process for finding the highest common divisor, we obtain, after a sufficient number of operations, a remainder in $y$ only. Now, if we *had known and substituted* the proper value of $y$, this remainder, which we will call R, should be $= 0$. Hence $R = f(y) = 0$ is a true equation. This equation is called the *final equation in y*.

Now, among the roots of this final equation in $y$ will be found all the compatible roots or values of $y$, and when they are substituted in the last preceding divisor placed equal to zero, they will give the corresponding values of $x$. That is, such will ordinarily be the result. But if the preceding divisor, upon the substitution of the values of $y$, becomes zero at once, so that we cannot obtain the corresponding values of $x$, we proceed to the next preceding one, which will be ordinarily of the second degree with regard to $x$, and give two values of $x$ to each one substituted for $y$. If this fails, we proceed to the divisor last before this, and so on.

But usually the substitution of the values of $y$, found from

the final equation in $y$, in the preceding divisor, will not at once reduce it to zero, but will give a polynomial in $x$ which will be a common divisor of the first members of the original equation after the value of $y$ has been substituted in them.

Now, this polynomial in $x$ should be equal to zero, because, being a common divisor of the first members of the original equations, it contains that factor of the form $x—a$, or the product of such factors, belonging to the compatible values of $x$, and probably other factors beside. This divisor, a $f(x)$, might, if we knew the factors composing it, be put under the form of $(x—a)f'(x)$, or $(x—a)(x—c)(x—d)f'(x)$, according to its nature, and of course the substitution for $x$ of the values, $a$, $c$, $d$, etc., would reduce it to zero.

This divisor, then, $f(x)= 0$, is a true equation, from which we ought to obtain the compatible values of $x$.

**Art. 11.** Having obtained the final equation in $y$, if by factoring its first member, or in any other way, we can solve it, we do so; substitute the roots for $y$ in the last preceding divisor, or in the one before that, as the case may require, obtain the values of $x$, and verify both by substituting them in the original equation.

**Art. 12.** *Foreign Roots.*—We will thus obtain all the *compatible* roots, but we may get also others. For if in preparing the dividends at any time, we have found it necessary to multiply any by $y$, or any $f(y)$, we may thus have introduced foreign values of $y$, which will appear among the roots of the final equation in $y$. This might have been done, for instance, to avoid having $y$ appear in any denominator of the quotient. For if we denote the first member of the first equation by $A$ and of the second equation by $B$, and by $Q$ the quotient of $A \div B$ and by $R$ the remainder, we shall have: $A = 0$, $B = 0$, and $A = BQ+R$, and from the next division: $B = RQ'+R'$, $R = R'Q''+R''$, etc. Now, since the equations are simultaneous and their first members have a common divi-

sor, the remainders $R'''$, $R''$, $R'$, etc., will on, substituting the values of $y$ and $x$ be found successively $= 0$, and finally $R = 0$. Now, if $Q$, or any quotient, should have $y$ in its denominator the substitution of its value in such denominator might reduce it to 0 and make $Q = \infty$, and then although $B = 0$, $BQ$ would not be $= 0$. The supposition on which the process is founded is that $A$ and $B$ are already (or have been made) whole with respect to $y$.

In this way foreign roots may have been introduced into the final equation. They may be detected by trial in the proposed equations and rejected.

Furthermore, it may happen that all the proper values may not be found by means of the final equation, since we may have suppressed some factors in the process for finding the H. C. D., which would reduce to zero on the substitution of the proper value of $y$. Such factors should be placed $= 0$ and the values of $y$ substituted, and the values of $x$ found from the resulting equations.

If the final remainder should be independent of $y$, it of course is not zero, and the equations have no compatible roots.

**Art. 13.** Mention has already been made of a solution of the final equation in $y$ when it was possible to detect its factors. Similarly when we can detect the factors of the first members of the proposed equations we may shorten the process. Suppose all the factors of each to have been discovered: they will be of two kinds, *common* and *those not common*. Among the common some may be altogether in $x$, some altogether in $y$ and some functions of both $x$ and $y$.

Likewise the same three species may exist among the factors not common.

Now any one of these factors being placed $= 0$ will satisfy the equation to which it belongs. Suppose we *first* consider those which are common to the two first members. Those in $x$ only will give a limited number of values for $x$ and any values whatever for $y$ provided they are finite;

those in $y$ only will give a limited member of values for $y$ and leave $x$ indeterminate, and those which are functions of $x$ and $y$ both will give an infinite number of sets of values for $x$ and $y$.

*Second.* Of those not common we cannot place two in $x$ only, or in $y$ only, equal to zero at the same time, for it would not be true unless one was equal to the other multiplied by some constant factor. And this is contrary to the supposition that we have already considered all that were common. There remain those not common and which contain both $x$ and $y$. To these we should apply the process for the H. C. D., and we will obtain a limited number of values for $x$ and also for $y$.

EXAMPLES.

1. Let the equations be $x^3-3yx^2+(3y^2-3y+1)x-y^3+y^2-2y=0$, and $x^2-2yx+y^2-y=0$.

First Division.

$$\begin{array}{l|l}
x^3-3yx^2+(3y^2-y+1)x-y^3+y^2-2y & \underline{x^2-2yx+y^2-y} \\
\underline{x^3-2yx^2+(y^2-y)x} & \quad x-y \\
\quad -yx^2+(2y^2+1)x-y^3+y^2-2y & \\
\quad \underline{-yx^2+2y^2x \qquad -y^3+y^2} & \\
\qquad\qquad x-2y &
\end{array}$$

This division was performed without preparation. So, likewise, with the

Second Division.

$$\begin{array}{l|l}
x^2-2yx+y^2-y & \underline{x-2y} \\
\underline{x^2-2yx} & x \\
\qquad y^2-y &
\end{array}$$

and $y^2-y=0$ is the final equation in $y$. Its roots are $y=0$ and $y=1$. Hence we have the systems

$$\begin{cases} y=0 \\ x=0 \end{cases} \text{ and } \begin{matrix} y=1 \\ x=2 \end{matrix}$$

2. Let the equations be $x^3+y^3=0$, and $x^2+xy+y^2-1=0$ No preparation will here be necessary.

First Division.

$$\begin{array}{r|l} x^3+y^3 & x^2+xy+y^2-1 \\ \cline{2-2} x^3+x^2y+y^2x-x & \quad x-y \\ \hline -yx^2-y^2x+x+y^3 & \\ -yx^2-y^2x-y^3+y & \\ \hline x+2y^3-y & \end{array}$$

Second Division.

$$\begin{array}{l|l} x^2+yx+y^2-1 & x+2y^3-y \\ \cline{2-2} x^2-yx+2y^3x & \quad x+2y-2y^3 \\ \hline (2y-2y^3)x+y^2-1 & \\ (2y-2y^3)x-4y^6+6y^4-2y^2 & \\ \hline \qquad 4y^6-6y^4+3y^2-1 & \end{array}$$

$4y^6-6y^4+3y^2-1$ and this placed $=0$ gives the final equation in $y$.

It is evident upon inspection that $y=1$ and $y=-1$ are roots of this equation, and the other four are $\pm\frac{1}{2}\sqrt{1\pm\sqrt{-3}}$.

$y=1$ and $y=-1$ give, upon substitution in the original equations, $x=1$ and $x=-1$, values to be expected, as the equations are symmetrical.

3. Let the equations be: $x^3+2yx^2+2y(y-2)x+y^2-4=0$ and $x^2+2xy+2y^2-5y+2=0$.

First Division.

$$\begin{array}{r|l} x^3+2x^2y+2y^2x-4yx+y^2-4 & x^2+2xy+2y^2-5y+2 \\ \cline{2-2} x^3+2x^2y+2y^2x-5yx+2x & x \\ \hline (y-2)x+y^2-4 & \end{array}$$

and this remainder may be factored thus, $(y-2)[x+y+2]$, and the factor $y-2$ laid aside.

Second Division.

$$\begin{array}{r|l} x^2+2yx+2y^2-5y+2 & x+y+2 \\ \cline{2-2} x^2+\ yx+2x \qquad\qquad & x+(y-2) \\ \hline (y-2)x+2y^2-5y+2 & \\ (y-2)x+\ y^2 \qquad -4 & \\ \hline y^2-5y+6 & \end{array}$$

which, placed $=0$, gives $y=2$ and $y=3$. $y=2$ gives $x=0$ and $x=-4$. $y=3$ gives $x=-1$ and $x=-5$

from the second equation, but upon trial with the first equation only the roots $x = -4$, $y = 2$, $x = -5$, $y = 3$, are found to be compatible roots.

The suppressed factor $y - 2$ gives $y = 2$; a value also found from the final equation.

The foregoing treatment of this subject is mainly taken from the excellent discussion of Elimination in Hackley's Algebra.

**Art. 14.** Labatie and Sarrus have perfected a method of elimination by which *foreign roots* are not introduced into the final equation. This mode is quoted by Hackley and by Todhunter in his Theory of Equations; but it is doubtful whether any advantage is gained over the simplicity and ease of trying the roots in the original equations and rejecting such as do not verify them.

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## CHAPTER V.

### Nature or General Theory of Equations.

**Art. 15.** An equation, as we know, is an algebraic expression of the equality of two quantities. (And before it can be solved must contain a single unknown quantity.) This statement is true even of an identical equation which is true for *any* value of the unknown quantity or quantities entering it; for the mind, in the act of attributing a value to any unknown quantity in the equation, may be supposed to regard that, for the time being, as the only one.

Now the two equal quantities, i. e., the two members, of every equation, may be placed in the first member leaving the second member zero; and the polynomial, after it has been arranged with reference to the descending powers of the unknown quantity, may be divided by the co-efficient of the highest power, and so at the same time may be the second member, placing the equation in the form

$$x^m + Px^{m-1} + Qx^{m-2} + Rx^{m-3} + \ldots + Tx + U = 0 \ldots (1)$$

This general form, which is often called *the reduced form*, has the co-efficient of $x^m$ unity and $P$, $Q$, $R$, $T$, etc., any quantities not transcendental; they may be algebraic or numerical, whole or fractional, positive or negative, rational, irrational, real, imaginary or zero. When any co-efficient is zero the corresponding power of the unknown quantity is usually absent. The equation is then *incomplete;* but when all the powers are present from the highest to the zero power the equation is *complete.*

The co-efficient of the zero power of the unknown quantity is called *the absolute term* of the equation.

**Art. 16.** The form $f(x) = x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0$ above described is the most convenient for examining the nature of equations, but many of the properties of equations which will be demonstrated are true when the equation has not been reduced to this form.

And, on the other hand, many properties will be demonstrated only of equations having *real* co-efficients and even of those having their co-efficients numbers.

**Art. 17.** *Every equation has at least one root.* Much ingenuity and mathematical skill have been used in demonstrating this proposition by algebraic analysis, but it seems unnecessary for it is almost if not quite axiomatic. Since an equation is an algebraic expression of the equality of two quantities, or of the fact that their difference is $= 0$, *there must be some quantity*, or value of the unknown, such that when its different powers have been multiplied by the appropriate co-efficients and the sum of all the products taken, the result shall be zero. *Otherwise there would be no equation;* the truth would not have been told by the algebraic expression.

The requisite value of the unknown quantity may be a real quantity or an imaginary expression; and it is called *a root of the equation.*

**Art. 18.** *Every equation of the mth degree has m roots and no more.*

We have just seen that every equation has at least one root, and we already know that an equation of the first degree has *one root;* also, that an equation of the 2d degree has *two roots*, and it is now to be proved that an equation of the $m$th degree has $m$ roots; that is, the number of roots is equal to the number of units in the exponent which shows the degree of the equation. It has been proved in Art. 9 that if $a$ is a root of an equation the second member of which is zero the first member will be exactly divisible by $x - a$.

Now suppose that $a$ is a root of the equation

$x^m + Px^{m-1} + Rx^{m-2} + Rx^{m-3} + — \ . \ + Tx + U = 0$, then we shall have $x^m + Px^{m-1} + Qx^{m-2} + \ldots Tx + U = (x - a)\ [x^{m-1} + P'x^{m-2} + \ldots + T'x + U'] \ldots (1)$

Now this can be satisfied by placing $x - a = 0$; and also by placing $x^{m-1} + P'x^{m-2} + — \ldots + T'x + U' = 0$, which is a new equation, and it also has at least one root.

Suppose that this root is $b$, then, as before, we have $x^{m-1} + P'x^{m-1} + \ldots + T'x + U' = (x—b)\ [x^{m-2} + P''x^{m-3} + Q''x^{m-4} + \ldots + T''x + U'']$ which can be satisfied by placing $x—b = 0$; and also, by placing $x^{m-2} + P''x^{m-3} + \ldots + T''x + U'' = 0$; and this is a new equation having at least one root, which may be called $c$, and when the corresponding factor $x—c$ is divided out we shall, as in the previous cases, have a new equation. The degree of this equation will be $m - 3$. Continuing this process until the original first member has undergone $m - 1$ successive divisions we shall have a quotient of the first degree, of the form $x - l$, which, placed equal to zero, gives an equation of the first degree, with one and but one root. Thus the total number of roots is $m$, and the continued product of the corresponding factors formed by subtracting each root from $x$ will be equal to the original first member, so that we shall have the equation

$f(x) = x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = (x-a)\ (x-b)$ $(x-c)\ (x-d) \ldots (x-l)\ \ldots (2)$

And there can be no more roots than $m$; for if there could be another and it were $k$, different from $a$, $b$, $c$, $d$.. $l$, there would be a factor $x-k$ which, multiplied into the product of all the others, would give for the first member a different polynomial, and one of a degree higher by unity; *hence k cannot be a root.*

This fact may also be seen thus: If $k$ is a value of $x$, let it be substituted in the continued product, $(x-a)(x-b)$ $(x-c)\ .\ .\ .\ .\ .\ (x-l) = 0$, and we derive $(k-a)(k-b)$ $(k-c)\ .\ .\ .\ .\ .\ (k-l)$, which cannot be zero, because none of the factors are zero; whereas, when a true root, as $a$, $b$, $c$, etc., is substituted, there will always be one factor which vanishes. Thus the theorem is seen to be true.

**Art. 19.** *Equal Roots.*—It may happen that one or more of the factors $x-a$, $x-b$, etc., shall be repeated, in which case the corresponding roots will appear as often in the equation; these are called *equal roots.* Thus, $(x-a)^3$ $(x-b)(x-c)^2 = 0$ is an equation of the 6th degree, which has *three equal roots*, $a$, and *two equal roots*, $c$.

It will be shown further on that when an equation has equal roots they may be discovered and the first member divided by the product of the factors belonging to them, thus *depressing* or reducing the degree of the equation. This operation is spoken of as "dividing out" the roots.

## COMPOSITION OF EQUATIONS.

**Art. 20.** When the roots of an equation are $a$, $b$, $c$, $d$, $e$, . . . . $l$, we have seen that

$$x^m + Px^{m-1} + Qx^{m-2} + Tx + U = (x-a)(x-b)(x-c)\ .\ .\ (x-l) \qquad \ldots\ldots (1)$$

Now, if the multiplications indicated in the second member be performed, the result will be as follows:

$$
x^m + x^{m-1}\left|\begin{array}{l} -a \\ -b \\ -c \\ -d \\ \dots \\ \dots \\ \dots \\ \text{to } m \text{ terms.} \end{array}\right. + x^{m-2}\left|\begin{array}{l} ab \\ +ac \\ +ad \\ \dots \\ +bc \\ +bd \\ \dots \\ \text{to } \frac{m(m-1)}{2} \text{ terms} \end{array}\right. + \text{etc.}, + x^{m-n}\left|\begin{array}{l} \pm abcd\ldots + \ldots + abcde\ldots \\ \pm abce\ldots \\ \pm abcf\ldots \\ \dots \\ \pm bcde \\ \pm bcdf\ldots \\ \dots \\ \text{to } \frac{m(m-1)\ldots(m-n+1)}{1.2.3\ldots n} \text{ terms} \end{array}\right. \quad \text{to } n \text{ letters.} \quad \text{to } m \text{ letters.}
$$

and since the equation of which this is the second member is identical, we have, from the principle of Indeterminates' Coefficients:

$P = -a-b-c-d\ .\ .\ -k-l$; or, $-P = a+b+c\ .\ .\ +k+l.$

$Q = ab+ac+\ .\ .\ +bc+\ .\ .\ +kl.$

$R = -abc-abd-\ .\ .\ -ikl$; or, $-R = abc+abd+\ .\ .\ +ikl.$

$S = abcd+abce+\ .\ .\ .\ gikl.$

. . . . . . . . . . . . .

. . . . . . . . . . . . .

$U = \pm abcde\ldots kl$; or, $\mp U = abcde\ldots kl.$

The quantities $a, b, c, d, e \ldots k, l$, all appearing with the negative sign, the product of an even number of them is plus and of an odd number minus, which accounts for the double sign wherever it appears, because in those cases the number of factors is not known.

From these results the following important relations of the roots of an equation to its coefficients are manifest, to wit:

First. The co-efficient of the second term (with its sign changed) is the algebraic sum of the roots.

Second. The co-efficient of the third term is the sum of the combinations of the roots in groups of two.

Third. The co-efficient of the fourth term (with its sign changed), is the sum of the combinations of the roots in groups of three; and so on.

Fourth. The co-efficient of the absolute term (with its sign changed when it is even numbered. i.e., when the degree of the equation is odd), is the continued product of the roots.

**Art. 21.** DEDUCTIONS.

Since the absolute term is the product of the roots it will be exactly divisible by any root; and, also, when there is no absolute term one of the roots is zero. Further, when there is no second term it is because the sum of the positive roots is exactly equal to the sum of the negative roots.

**Art. 22.** When the roots of an equation are all positive, the terms will be alternately positive and negative; because the product of an even number of negative terms is plus and of an odd number is minus.

**Art. 23.** Since the first member of an equation of which the second $=0$ is composed by multiplying together the factors $(x-a)$, $(x-b)$, etc., it will have $m$ factors or divisors of the 1st degree; and since any two of them may be multiplied together, giving a factor of the 2d degree, any three giving a factor of the third degree, and so on, there will be $\frac{m(m-1)}{2}$ divisors of the 3d degree, $\frac{m(m-1)(m-2)}{1.2.3}$ divisors of the 4th degree, and so on.

**Art. 24.** When $a$, $b$, $c$, etc., are the roots of an equation that equation is $(x-a)(x-b)(x-c)\ldots.(x-i)=0$. Suppose the roots of an equation are 1, 2, 3, 4: the equation is $(x-1)(x-2)(x-3)(x-4)=x^4-10x^3+35x^2-50x+24=0$.

EXAMPLES.

1. Form the equation whose roots are 3, 7 and $-6$.

2. Form the equation in which the roots are 9, 5, $-1$, and $-3$.

3. What is the equation of which the roots are $-3$, $2+\sqrt{-1}$ and $2-\sqrt{-1}$? Ans. $x^3-x^2-7x+15=0$.

**Art. 25.** Since in the reduced equation $U=abcde\ldots l$, $T=abcde..k+abcde..h+abcde..g, +$ etc., where the terms of the value of $T$ are composed of $m-1$ letters each, if we divide the latter by the former we get

$$\frac{T}{U}=\frac{abcde\ldots h}{abcde\ldots hl}+\frac{abcde\ldots g}{abcde\ldots gl}+\text{etc.},=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\text{etc.}$$

Therefore, the co-efficient of the last but one divided by the absolute term is the *sum of the reciprocals of the roots.*

In the same way it may be shown that $\frac{S}{U}=\frac{1}{ab}+\frac{1}{ac}+\frac{1}{ad}+\frac{1}{bc}+$ etc., wherein $S$ is the coefficient of $x^2$.

**Art. 26.** *If the coefficients of an equation are whole numbers, no root can be an exact fraction.*

For, suppose $\frac{a}{b}$, an irreducible fraction, to be a root of $x^m+Px^{m-1}+Qx^{m-2}+\ \ldots\ +Tx+U=0$; then, since $\frac{a}{b}=x$, $\frac{a^m}{b^m}+P\frac{a^{m-1}}{b^{m-1}}+\ldots\ldots+T\frac{a}{b}+U=0$; $\therefore \frac{a^m}{b}=-Pa^{m-1}-Qa^{m-2}b-Ra^{m-3}b^2-\ \ldots\ -Tab^{m-2}-Ub^{m-1}$. The second member is whole, and $\frac{a^m}{b}$ is a fraction, since $b$ is supposed to be prime to $a$, and therefore to $a^m$. Here, then, is an absurdity of a fraction equal to a whole number, which establishes the proposition.

**Art. 27.** *The imaginary roots of an equation enter by pairs, when the coefficients are real; and if the coefficients are rational, all roots which are not rational enter by pairs.*

Imaginary, or impossible, roots, as they are sometimes called, are cases of the general form, $a+\sqrt{-b^2}$ and $a-\sqrt{-b^2}$. These give the factors $(x-a-\sqrt{-b^2})(x-a+\sqrt{-b^2})=x^2-2ax+a^2+b^2$. The product of the imaginary roots is $(a+\sqrt{-b^2})(a-\sqrt{-b^2})=a^2+b^2$; or, $(a+\sqrt{-b})(a-\sqrt{-b}=a^2+b$; both real, rational and positive. The product of irrational but real roots: $(a+\sqrt{b})(a-\sqrt{b})=a^2-b$, which is real but not necessarily positive. If either of these be substituted in an equation for $x$, it is apparent that the results will be partly real and partly imaginary, unless some of the coefficients could furnish the necessary factor to make the imaginary quantities disappear from the product. But the coefficients in this article are supposed to be real. Consequently there must be another imaginary root of the proper form, to cause the product to be real. If, now, a third imaginary root should enter into the composition of the equation, a fourth, and of the necessary form, must enter to keep the product real. *There cannot be, therefore, an odd number of imaginary roots.*

If the coefficients are further supposed to be all *rational*, it is evident by the same course of reasoning that all irrational roots must enter by pairs.

**Art. 28.** *Hence every equation of an odd degree has at least one real root*, with a sign different from that of the absolute term. The imaginary roots are of the form $a+\sqrt{-b^2}$ and $a-\sqrt{-b^2}$, or else $a+\sqrt{-b}$ and $a-\sqrt{-b}$, and their products result in the sums of positive quantities. And this positive sum is a factor of the absolute term and exercises no influence on the sign of that term. And so of the product of all of the imaginary pairs. This leaves the one real root to give sign to the absolute term which, of course, is the opposite of its own. (See Art. 20.)

It is also true that every equation of an odd degree having *rational* co-efficients will have at least one *rational* root; the sign may or may not be the same as that of the absolute term.

**Art. 29.** *Every equation of even degree and having real co-efficients, with its absolute term negative, will have at least two real roots, one positive and the other negative.* The products of the pairs of imaginary roots will exert no influence on the sign of the absolute term, and if *all the roots were imaginary the absolute term would be positive,* but as it is not positive there must be at least two real roots and such that their product will be negative, they, therefore, must have different signs.

**Art. 30.** *Every equation will have an even number of real positive roots if the absolute term is positive; and an odd number of such roots if the absolute term is negative.*

First. When the degree is even and the absolute term positive. The degree being even the number of the absolute term is odd, and, therefore, it is the continued product of the roots just as it stands. In this case that product is positive. If there are any imaginary roots, the quadratic factors belonging to the pairs will exert no influence on the sign of the absolute term. The total number of roots being even, the number of real roots must be 0 or even. Now the product of the real roots must be positive and if there is any real root negative there must be another one negative to neutralize the influence of the sign, otherwise the sign of the absolute term would be changed. Hence the number of the real positive roots is even, which proves the theorem for this case.

It is apparent that the number of *negative* real roots would also be even.

Second. When the degree is even and the absolute term negative. Here, also, the absolute term, as it stands, is the product of the roots; and if there are imaginary roots they exert no influence on the sign. The number of real roots is even, and since the product is negative there must be at least one which is negative. This may be considered as set aside for the moment. There now remain for consideration an odd number of real roots, whose product must be positive, and if there is among these a negative

root, there must also be another of the negative sign to neutralize its effect; in other words, if there are any negative roots among those now being considered, there must be an even number of them; consequently, the *number of real positive roots is odd*.

THIRD. When the degree is odd and the absolute term positive. In this case the sign of the absolute term must be changed to give the continued product of the roots; that is, the product is negative. We know (Art. 28), that for such an equation there is one real and negative root. Let that be set aside as necessary to change the sign of the absolute term. The total number of roots remaining is even, and the number of real roots remaining is even. Moreover the product of these remaining roots must be positive and consequently if there are *among these* any negative roots there must be an even number and therefore *the number of real positive roots must be even*.

FOURTH. When the degree is odd and the absolute term negative (By Art. 28) there is one real positive root. The total number remaining is even, and the number of real roots remaining is even; but their product must be positive as the real root set aside is positive; consequently among these remaining real roots if there are any negative roots there must be an even number of them, likewise an even number of positive real roots, which, with the one set aside, makes the *number of real positive roots odd*. Thus the theorem is established.

**Art. 31.** *If the signs of the alternate terms of an equation be changed, the roots of the new equation will be the same as those of the former equation but with opposite signs.*

Let the equation be $x^m+Px^{m-1}+Qx^{m-2}+\ldots.+Tx+U=0\ldots$ (1) If we change the alternate signs, beginning with the second, we have $x^m-Px^{m-1}+Qx-^{m-2}\ldots\mp Tx\pm U=0\ldots.(2)$., and beginning with the first, we have $-x^m+Px^{m-1}-Qx^{m-2}+\ldots.\pm Tx\mp U=0\ldots.(3)$, which equations, (2) and (3), are merely one and the same. Now suppose $+a$ to be a root of (1) and to be substituted in it for $x$.

The result will be $a^m+Pa^{m-1}+Qa^{m-2}+\ldots+Ta+U=0$. (4). Now if $-a$ is a root of (2) and (3) it must, on substitution in the one or in the other (as may be suitable, for they are merely two forms of the same thing), give eq. (4).

When $m$ is even use (2) and when $m$ is odd use (3). Therefore $-a$ is a root of (2) or else of (3). Hence the principle. Changing the signs of all the terms would not affect the roots, since it would simply be multiplying both members of the equation by $-1$.

### DESCARTES' RULE.

**Art. 32.** *No equation can have more positive roots than there are* VARIATIONS *in the signs of its terms, nor more negative roots than there are* PERMANENCES *of those signs.*

To demonstrate this assume the equation $x^m \pm Px^{m-1} \pm Qx^{m-2} \pm Rx^{m-3} \pm \ldots \pm Tx \pm U=0$; in which the signs come in any order that may be prescribed. Now suppose that we introduce one more positive root, which will be done by multiplying by $x-a$, and note the effect on the signs. The product will be

$$\begin{array}{r|r|r|r|l} x^{m+1}\pm P & x^m \pm Q & x^{m-1}\pm R & x^{m-2}\pm\ldots\pm U & x \\ -a & \mp Pa & \mp Qa & \mp T & \mp Ua=0. \end{array}$$

Now so long as the co-efficient in the upper line is greater than the one in the lower line it will determine the sign of the total co-efficient of that term; if we suppose then in the first case, that all the upper were greater than those below we would have the same number of variations and permanences as in the original equation, but having to come down at last to $\mp Ua$, there is one more variation than in the original equation. If the lower co-efficients are all greater than those above they will give sign to the terms; but the signs from the second toward the right, being always the opposite of the signs of the original first member, the number of changes of sign and of permanence, or repetition of sign, will be the same. But one more variation was introduced when we descended at the second term.

When a co-efficient in the lower line is affected with a sign contrary to the corresponding one above and is also

greater than that above, there is a change from a permanence of sign to a variation, for the lower co-efficient gives sign to the term, and we know that it is different from that of the preceding term above which is here supposed to be the same as that of the co-efficient above in this term. Hence each time we descend to the lower line in order to determine the sign there is a variation which is not found in the original equation, and if, after descending, we remain in the lower line throughout, the number of permanences and variations of sign henceforth will be the same as in the given equation because the signs are always the opposite of those above. If we ascend again to the upper line, we might make either a permanence or a variation; but suppose the worst, and that always there would be a permanence, it would merely offset the variation gained in coming down, and it will be necessary to come down at last, making a variation at that time. Therefore, the effect has been to produce one more variation than the original equation had; and so it would be upon the introduction of every positive root.

Similar reasoning would show that the multiplication by the factor $x+a$, belonging to a negative root, would necessarily introduce a *permanence* of sign. And since the introduction of every positive root brings a variation, and the introduction of every negative root brings a permanence, the Rule of Descartes is shown to be true.

**Art. 33.** *When the roots are all real* the number of positive roots will be the number of variations, and the number of negative roots the number of permanences.

Suppose that the degree of the equation was $m$; then, the complete number of terms being $m+1$, and $n$ representing the number of variations and $p$ the number of permanences, $m=n+p$.

Again, suppose that $k =$ the number of positive roots and $r =$ the number of negative roots. We shall have: $m = k+r$; hence $n+p = k+r$, and $n-k = r-p$. Now,

by Descartes' Rule, $k$ cannot be $>n$; nor can it be less, because that would make, in the second member, $r>p$, which the Rule forbids. Therefore $n=k$ and $p=r$.

### DE GUA'S CRITERION.

**Art. 34.** *If a term of an equation is absent between two terms having like signs, there are two imaginary roots.*

The absent term having 0 for a co-efficient, we have a right to supply it either as $+0$ or $-0$. Suppose the order of signs to be:

$+ + - \ 0 - + - -$, and for 0 writing $+$ or $-$,

we have: $+ + - + - + - -$ and

$+ + - - - + - -$.

In the upper line are 5 variations, 2 permanences.

In the lower, 3 variations, 4 permanencies.

Now, if all the roots are supposed to be real, there will by the first arrangement be 5 positive roots and 2 negative; by the second, 3 positive and 4 negative. There are, then, two roots which have changed about, being in one case positive and in the other negative. But both suppositions being legitimate, we have two real roots, which are both positive and negative, which absurdity shows them to be imaginary. Where the terms between which the zero term is found have contrary signs, we can predicate nothing about the nature of the roots, because in that case the number of variations and permanences will be the same, whether we suppose the absent term to be positive or negative.

### EXAMPLES.

How many imaginary roots in—

$x^5+x^3-2x^2+2x-1=0$?

$x^4-x^3+6x^2+24=0$?

$x^4-2x^2+6x+10=0$?

# CHAPTER IV.

## Transformation of Equations.

**Art. 35.** The changing the form of an equation, and yet preserving the equation is an operation not only allowable but often of the greatest convenience.

We have seen already (Art. 31), that the signs of the roots of an equation may all be changed by changing the signs of the alternate terms; that is, the changing of the signs of the terms in this manner gives another equation whose roots are numerically the same, but have opposite signs to those of the first equation.

**Art. 36.** *To transform an equation into another equation whose roots shall be some multiple of the roots of the first.*

Let the equation be $x^m + Px^{m-1} + Qx^{m-2} + \dots + Tx + U = 0, \dots (1)$ and suppose its roots to be $a$, $b$, $c$, etc. It is required to produce an equation of which the roots shall be $ka$, $kb$, $kc$, etc.

Make $y = kx \therefore x = \frac{y}{k}$, and this, substituted for $x$, gives

$$\frac{y^m}{k^m} + P\frac{y^{m-1}}{k^{m-1}} + Q\frac{y^{m-2}}{k^{m-2}} + \dots\dots + T\frac{y}{k} + U = 0, \dots . (2).$$

Multiplying by $k^m$, we get:

$$y^m + Pky^{m-1} + Qk^2y^{m-2} + \dots + Tk^{m-1}y + Uk^m = 0 \dots\dots . (3).$$

In this equation, since $y = kx$, the roots are $ka$, $kb$, $kc$, etc.

**Art. 37.** This transformation leads to one of the most important, which is

*To clear an equation of fractions and yet keep the co-efficient of the highest power unity.*

Let $x^m + \frac{P}{k}x^{m-1} + Qx^{m-2} + \frac{R}{g}x^{m-3} +$ etc., $= 0$ be an equation having fractional co-efficients, and place $y = gkx$, that is, equal to $x$ multiplied by the least common multiple of the denominators. Then

$$\frac{y^m}{k^m g^m}+\frac{Py^{m-1}}{k^m g^{m-1}}+\frac{Qy^{m-2}}{k^{m-2}g^{m-2}}+\frac{Ry^{m-3}}{k^{m-3}g^{m-2}}+\text{ etc.},=0.$$

Multiply by $k^m g^m$, and we have

$$y^m+gPy^{m-1}+k^2g^2Qy^{m-2}+k^3g^2Ry^{m-3}+\text{ etc.},=0.$$

This is an equation of the reduced form, wherein (if the roots of the original equation are $a$, $b$, $c$, etc.) the roots are $kga$, $kgb$, $kgc$, etc.

If the denominators are numbers we may obtain a transformed equation of greater convenience by assuming for $kg$ a number less than the L.C.M. of the denominators, but which shall be such a product of prime factors of the denominators as shall secure, after the substitution, an entire quotient in each co-efficient. This will be a matter of inspection and discretion to be used in each example.

For instance take the equation

$x^4-\frac{5}{6}x^3+\frac{5}{12}x^2-\frac{7}{150}x-\frac{13}{9000}=0$. Here 9000 is the L.C.M. of the denominators, but the 3d and 4th powers of 9000 are inconveniently large.

But $6=2\times3$; $12=2^2\times3$; $150=2\times3\times5^2$, and $9000=2^3\times3^2\times5^3$.

Suppose that in the example we make $y=2\times3\times5x=30x$; we shall obtain:

$$\frac{y^4}{2^4.3^4.5^4}-\frac{5y^3}{6.2^3.3^3.5^3}+\frac{5y^2}{12.2^2.3^2.5^2}-\frac{7y}{150.2.3.5}-\frac{13}{9000}=0.$$

Now, the denominator $9000=2^3.3^2.5^3$ is the most difficult one to provide for, and yet it will disappear when we multiply by $2^4.3^4.5^4$. The result will be:

$$y^4-5.5y^3+5.3.5^2y^2-7.2^2.3^2.5y-13.2.3^2.5=0;\text{ or,}$$
$$y^4-25y^3+375y^2-1260y-1170=0.$$

If the roots of this equation can be found, those of the first will result from the relation $y=30x$.

EXAMPLE 2.

$x^3-\frac{7}{3}x^2+\frac{11}{36}x-\frac{25}{72}=0$. If we make $y=2\times3x$, we get $y^3-14y^2+11y-75=0$.

EXAMPLE 3.

$$x^5-\frac{13}{12}x^4+\frac{21}{40}x^3-\frac{32}{225}x^2-\frac{43x}{600}-\frac{1}{800}=0.$$

$12 = 2^2.3$

$40 = 2^3.5$

$225 = 3^2.5^2$

$600 = 2^3.3.5^2$

$800 = 2^5.5^2$. The prime factors being 2, 3 and 5 it might appear that 2.3.5 would be a proper multiplier for $x$, but on trial we would find at the third term $\frac{21.2^5.3^5.5^5.y^5}{2^3.5.2^3.3^3.5^3}=\frac{21.3^2.5y^5}{2}$, an irreducible fraction. But if we use the product $2^2.3.5$ we shall obtain $y^5-65y^4+1,890y^3-30,720y^2-928,800y-972,000=0$ in which $x=\frac{1}{60}y$.

If the equation has *the co-efficient of the highest power different from unity;* divide through by that co-efficient and then proceed as before.

Suppose $3y^3-qy+r=0$, or $y^3+\frac{0}{3}y^2-\frac{q}{3}y+\frac{r}{3}=0$.

Put $x=3y$. $\therefore \frac{x^3}{27}-\frac{qx}{9}+\frac{r}{3}=0$; $\therefore x^3-3qx+qr=0$.

**Art. 38.** From equation 3, Art. 36, we see that if the second term of an equation is exactly divisible by $k$, the third term by $k^2$, the fourth by $k^3$, etc., its roots will have a common divisor $k$.

And any equation may be transformed into another of which the roots are $\frac{1}{k}$ of those of the former by dividing the second term by $k$, the third by $k^2$, the fourth by $k^3$, etc. This would give at once the result of making the multiplier $\frac{1}{k}$ instead of $k$ in the transformation of Art. 36.

For an example take the equation

$x^3-8x^2-5x+84=0$. . .(1) and let the second co-efficient be divided by 2, and the succeeding co-efficients respectively

by 4 and 8. We get $x^3-4x^2-\frac{5}{4}x+\frac{84}{8}=0\ldots(2)$. The roots of equation (1) are 7, —3 and 4, and $\frac{7}{2}, -\frac{3}{2}$ and 2 will verify eq. (2).

**Art. 39.** *To transform an equation into another, the roots of which shall be the reciprocals of those of the first.*

Substitute $\frac{1}{y}$ for $x$ in the equation

$x^m+Px^{m-1}+Qx^{m-2}+\ldots+Tx+U=0$, and the result is $\frac{1}{y^m}+\frac{P}{y^{m-1}}+\frac{Q}{y^{m-2}}+\ \ldots\ldots+\frac{T}{y}+U=0$; whence, by clearing and reversing the order of the terms and dividing by $U$, $y^m+\frac{T}{U}y^{m-1}+\frac{S}{U}y^{m-2}+\ldots+\frac{Q}{U}y^2+\frac{P}{U}y+\frac{1}{U}=0$. in which, if the original roots were $a$, $b$, $c$, these roots are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, etc.

**Art. 40.** If any term is wanting in the given equation, there will be one wanting in the transformed equation at the same distance from the last as the other was from the first term. If the original equation is wanting in the second term, the one next to the last will be wanting in the transformed one, because the latter coefficients are equal to the former divided by $U$, and $\frac{0}{U}=0$.

**Art. 41.** If an equation be transformed by making $x=\frac{1}{y}$, and the transformed equation should have the coefficients identical with those of the given equation, but in reversed order, the two equations are one and the same. This is evident upon sight, *and therefore their roots must be the same.* If the roots of the original equation were $a$, $b$, $c$, $d$, etc., the roots of the transformed one must also be $a$,

$b$, $c$, etc. But we know that the roots are also $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, etc.; hence the roots of both are: $a$, $\frac{1}{a}$, $b$, $\frac{1}{b}$, $c$, $\frac{1}{c}$, etc. $x^4-px^3+qx^2-px+1=0$, $x^4+qx^2+1=0$, $x^4+1=0$ in which the coefficients are: $1 -p +q -p +1$, $1 +q +1$, $1 +1$, are of the kind whose roots are of the form $a$ and $\frac{1}{a}$.

**Art. 42.** If we have an equation of an odd degree, or one of an even degree without its middle term, and the signs of the corresponding terms, counting from first to last, and from last to first, are opposite, the roots will also be of the form $a$, $\frac{1}{a}$. Because, if we obtain the transformed equation, and then change the signs throughout, we do not affect the roots at all (Art. 31), and yet it becomes identical with the original equation, and must therefore have the same roots. For example, let

$$x^3-px^2+px-1=0.$$

Substituting $\frac{1}{y}$ for $x$, we have, after clearing,

$$1-py+py^2-y^3=0; \text{ or, } y^3-py^2+py-1=0.$$

Equations whose roots are of the form $a$, $\frac{1}{a}$, $b$, $\frac{1}{b}$, etc., are called *recurring equations.*

**Art. 43.** *A recurring equation of an odd degree* must have 1 for a root when the absolute term is $-1$, and $-1$ for a root when the absolute term is $+1$; because these numbers being substituted for $x$ will satisfy it.

Let $x^5-px^4+qx^3-x^2+px-1=0$, and substitute $+1$ for $x$, we get $1-p+q-q+p-1=0$. The other roots (Art. 42) will be of the form $a$, $\frac{1}{a}$, $b$, $\frac{1}{b}$, etc.

**Art. 44.** *To transform an equation into another of which the roots shall be the squares of the roots of the first.*

Let us assume, for convenience, that in the first member of the equation the even numbered terms are negative, and transpose all the negative terms to the second member. We shall have:

$$x^m+qx^{m-2}+sx^{m-4}+\text{ etc.}, =px^{m-1}+rx^{m-3}+\text{ etc.}$$

Square both members, and we have:

$$x^{2m}+2qx^{2m-2}+(q^2+2s)x^{2m-4}+\text{ etc.},=p^2x^{2m-2}+2prx^{2m-4}+\text{ etc.}$$

And therefore $x^{2m}+(2q-p^2)x^{2m-2}+(q^2+2s-2pr)x^{2m-4}+$ etc., $=0$. Now this is a true equation, as we have a right to square both members. Let $y=x^2$, and substitute in the last equation; the result is: $y^m+(2q-p^2)y^{m-1}+(q^2+2s-2pr)y^{m-2}+$ etc., $=0$, an equation whose roots are the squares of those of the first.

EXAMPLE.

Let $x^3+3x^2-6x-8=0$. In this by transposition we have $x^3-6x=8-3x^2$, and by squaring, $x^6-12x^4+36x^2=9x^4-48x^2+64$; whence $x^6-21x^4+84x^2-64=0$, and placing $y=x^2$, $y^3-21y^2+84y-64=0$. By trial we find $-1$ is a root of the given equation, and "dividing it out," we find the others to be $-4$ and 2. Squaring these, we get 1, 16 and 4, which are roots of the new equation and will verify it.

**Art. 45.** *To transform an equation into another whose roots shall be greater or less than those of the first equation by any given quantity.*

First place, if necessary, the equation in the reduced form:

$$x^m+Px^{m-1}+Qx^{m-2}+\ldots+Tx+U=0, \ldots (1)$$

Let $x'$ be any given quantity, and make $y\pm x'=x$. The new equation in $y$ will have its roots greater or less than the roots of the original one by $x'$. Let us use the $+$ sign only; the results of substituting $y-x'$ would only differ in sign at the appropriate places. Substituting $y+x'$ for $x$, we obtain:

$$(y+x')^m+P(y+x')^{m-1}+Q(y+x')^{m-2}+\ldots+T(y+x')+U=0.$$

Now, develop by the Binomial Theorem, and arrange ac-

cording *to the ascending powers of the unknown quantity* $y$, (which is done merely for subsequent convenience), there will result:

$$
\left.\begin{array}{l}
x'^m \\
+Px'^{m-1} \\
+Qx'^{m-2} \\
+\dots\dots \\
+\dots\dots \\
+Tx' \\
+U
\end{array}\right| y^0
\left.\begin{array}{l}
+mx'^{m-1} \\
+P(m-1)x'^{m-2} \\
+Q(m-2)x'^{m-3} \\
+\dots\dots\dots\dots \\
+\dots\dots\dots\dots \\
+T
\end{array}\right| y
\left.\begin{array}{l}
+\dfrac{m(m-1)}{1.2}x'^{m-2} \\
+P\dfrac{(m-1)(m-2)}{1.2}x'^{m-3} \\
+Q\dfrac{(m-2)(m-3)}{1.2}x'^{m-3} \\
+\dots\dots\dots\dots\dots\dots
\end{array}\right| y^2 + \dots
\left.\begin{array}{l}
+mx' \\
+P
\end{array}\right| y^{m-1} + y^m = 0 \dots\dots (2)
$$

And this is the required transformed equation, but with the usual order of the terms reversed. If $P'$, $Q'$, etc., rep-

resent the values of the co-efficients from $y^{m-1}$ down, and the usual descending order be resumed, the equation will be:

$$y^m + P'y^{m-1} + Q'y^{m-2} + \ldots + T'y + U' = 0, \ldots (3)$$

**Art. 46.** A method of arriving at the values of the transformed co-efficients, $P'$, $Q'$, $R'$, etc., which is preferred by some as being shorter, is as follows:

*Divide the first member of the equation to be transformed by $x$ minus the difference between the old and new roots, the remainder will be the new absolute term; divide, by the same, this quotient, and the remainder will be the co-efficient of the first power of the new unknown; divide, by the same, the last quotient obtained and the remainder will be the next co-efficient in order, and so on to the last co-efficient.*

Equation (3) of the preceding article was obtained from eq. (1) by making $x = y + x'$; consequently if we make in (3) $y = x - x'$ we shall simply go back to (1), and the first members will give an *identical equation:* $(x - x')^m + P'(x - x')^{m-1} + \ldots + T'(x - x') + U' = x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0$.

If we divide both members of this equation by any quantity the quotients must be the same; dividing the first member by $x - x'$, the quotient is $(x - x')^{m-1} + P'(x - x')^{m-2} +$ etc., and the remainder $U'$, which is the absolute term of the transformed equation. Consequently, had we divided the second number, which is *the first member of the original equation* by $x - x'$, we would have obtained a quotient and remainder identical with these.

Dividing this quotient by $x - x'$ we have as a remainder $T'$, and would have had it had we divided the quotient in the division of the original first member, and so on successively we would obtain all the co-efficients of the transformed equation.

### EXAMPLE.

Transform the equation $x^3 - 2x^2 + 3x - 4 = 0$ into one of which the roots shall be less by 1.7.

First Operation.

$$\begin{array}{l|l} x^3 - 2x^2 + 3x - 4 & x - 1.7 \\ \underline{x^3 - 1.7x^2} & \overline{x^2 - 0.3x + 2.49} \\ \quad -.3x^2 + 3x - 4 & \\ \quad \underline{-.3x^2 + .51x} & \\ \qquad 2.49x - 4 & \\ \qquad \underline{2.49x - 4.233} & \\ \qquad\qquad .233 & \end{array}$$

$.233 =$ absolute term of new equation.

2d Operation.

$$\begin{array}{l|l} x^2 - 0.3x + 2.49 & x - 1.7 \\ \underline{x^2 - 1.7x} & \overline{x + 1.4} \\ \quad 1.4x + 2.49 & \\ \quad \underline{1.4x - 2.38} & \\ \qquad +4.87 & \end{array}$$

$+4.87 =$ co-efficient of $y$ in new equation.

3d Operation.

$$\begin{array}{l|l} x + 1.4 & x - 1.7 \\ \underline{x - 1.7} & \overline{1} \\ \quad 3.1 & \end{array}$$

$3.1 =$ co-efficient of $y^2$ in new equation.

Hence the transformed equation is $y^3 + 3.1y^2 + 4.87y + .233 = 0$.

These three divisions may be performed more expeditiously by

SYNTHETICAL DIVISION.

**Art. 47.** Synthetical Division is a short mode of dividing polynomials wherein we make use of the co-efficients only. Let us perform the three divisions above in this manner.

$$\begin{array}{l|l|l|l} 1 - 2 + 3 - 4 & 1 - 1.7 & & \\ \underline{1 - 1.7} & 1 - 0.3 + 2.49 & 1 - 1.7 & \\ \quad -0.3 + 3 - 4 & \underline{1 - 1.7} & 1 + 1.4 & 1 - 1.7 \\ \quad \underline{-0.3 + .51} & \quad 1.4 + 2.49 & \underline{1 - 1.7} & 1 \\ \qquad 2.49 - 4 & \quad \underline{1.4 - 2.38} & +3.1 & \\ \qquad \underline{2.49 - 4.233} & \qquad +4.87 & & \\ \qquad\qquad +.233 & & & \end{array}$$

The successive remainders are $+0.233$, $+4.87$ and

$+3.1$ which are the co-efficients of $y^2$, $y$, $y^0$ respectively, and hence the transformed equation is $y^3+3.1\,y^2+4.87y+0.233=0$.

In this skeleton division we have simply applied the rule for division of polynomials. Since the first term of the divisor is unity, *the first term of every quotient must be the same as the first term of the dividend.*

To get the second term of the quotient we have multiplied the second term of the divisor by the first term of the quotient and then *subtracted* this result from the second term of the dividend. And this difference divided by 1 (the first term of the divisor) would give itself as the second term of the quotient. Now we would have arrived at the same result if we had at once changed the sign of the second term of the divisor, multiplied this by the first term of the quotient (the same as the first term of the dividend) and had *added* this product to the second term of the dividend.

Then this *algebraic sum* would have been the second term of the quotient.

The third would, in a similar way, have been obtained by changing the sign of the second term of the divisor, multiplying it by the second term of the quotient and *adding* the product to the third term of the dividend. And so proceeding until the first remainder was found, which would give the absolute term. In a precisely similar manner an expeditious division may be made of the successive quotients until all the co-efficients of the transformed equation shall have been obtained.

It will be observed that these operations are performed without making any use of unity as the first term of the divisor. We have then for the synthetical division used in this transformation the following

RULE.

*Write the co-efficients of the first member of the given equation, in the reduced form, in thier order with their proper*

*signs. Change the sign of the quantity which is the difference between the old and new roots and call it the multiplier.*

*The first term of the dividend is the first term of the quotient. Multiply it by the multiplier and add the product to the second term of the dividend, the result will be the second term of the quotient; then multiply this by the multiplier and add to the third term of the dividend, and the result will be the third term of the quotient. So proceed until the first remainder is obtained. It will be the absolute term required.*

*Using the same multiplier, treat in the same way the successive quotients and obtain by the successive remainders, all the coefficients up to that of the power next to the highest of the unknown quantity.* The first coefficient is unity.

Let us apply this rule to to the last example.

| | | | | | |
|---|---|---|---|---|---|
| | 1 | —2 | +3 | —4 | \| +1.7 |
| | | 1.7 | —0.51 | +4.233 | |
| 1st quotient, | 1 | —0.3 | +2.49, | +0.233 | 1st remainder. |
| | | 1.7 | +2.38 | | |
| 2d quotient, | 1 | +1.4, | +4.87 | | 2d remainder. |
| | | 1.7 | | | |
| 3d quotient, | 1, | +3.1 | | | 3d remainder. |

And the equation as before, is $y^3+3.1y^2+4.87y+0.233=0$.

**Art. 48.** This discussion of Synthetical Division is here adduced for the information of the student, and to be used in similar transformations, and especially in the operations of Horner's Method of approximating to the roots of numerical equations. But it is thought best for him to defer its use until he is familiar with the principles of the main operations which it is intended to facilitate.

Thus it would, on first going over this subject, probably be better for the student to obtain the various coefficients of the transformed equation by the ordinary division of polynomials.

In fact, where the numbers are not large and do not have to be raised to very high powers it will be as well to make the plain and simple substitutions and to perform the indicated operations, as follows:

Resuming the same example: to find an equation whose roots are 1.7 less than those of the equation

$$x^3-2x^2+3x-4=0.$$

Place $x=y+1.7$ $\quad (1.7)^2=2.89;\ (1.7)^3=4.913$

$$\begin{array}{rlrrrr} x^3 &= (y+1.7)^3 = y^3 & +\ 5.1y^2 & +\ 8.67y & +\ 4.913 & \\ -2x^2 &= -2(y+1.7)^2 = - & 2y^2 & -\ 6.80y & -\ 5.780 & \\ +3x &= \quad 3(y+1.7) = & & +\ 3.00y & +\ 5.100 & \\ -4 &= & & & -\ 4 & \\ \hline & \quad y^3 & +3.1y^2 & +\ 4.87y & +\ 0.233 & = 0 \end{array}$$

EXAMPLES.

1. Find an equation whose roots are less by 1 than those of $x^3-7x+7=0$. Ans. $y^3+3y^2-4y+1=0$.

2. Find the equation whose roots are greater by $e$ than those of $x^3-px^2+qx-r=0$.

Ans. $y^3-(3e+p)y^2+(3e^2-2pe+q)y-(e^3+pe^2-qe+r)=0$.

3. Find the equation whose roots are greater by 2 than those of $x^4-2x^3+5x^2+4x-8=0$. Ans. $y^4-10y^3+41y^2-72y+36=0$.

4. Find the equation whose roots are greater by 1 than those of $x^4-5x^2-6x-2=0$. Ans. $y^4-4y^3+y^2=0$.

As $y$ is twice a factor of every term of this transformed equation, let us "divide out" $y^2$, and we have $y^2-4y+1=0$, whose two roots are $2+\sqrt{3}$ and $2-\sqrt{3}$, and as the four roots are 0, 0, $2+\sqrt{3}$, $2-\sqrt{3}$, if we subtract 1 from each we get the roots of the given equation, $-1$, $-1$, $1+\sqrt{3}$, $1-\sqrt{3}$.

**Art. 49.** *To transform an equation into another wanting the second or any particular term.*

From equation (2) of Art. 45 we see that the coefficient of $y^{m-1}$ (which is the second term in the usual arrangement) is $mx'+P$. Now, since $x'$ is entirely arbitrary, we can give it such a value that $mx'+P=0$, $\therefore\ x'=-\dfrac{P}{m}$; that

is, minus the coefficient of the second term divided by the exponent which denotes the degree of the equation to be transformed. All we have to do is to substitute for $x$ the quantity $y-\frac{P}{m}$. The equation resulting will have roots greater by $\frac{P}{m}$ than the original roots.

To cause the third term to be absent from the new equation, we must place the coefficient of $y^{m-2}$, which is $\frac{m(m-1)}{1.2}x'^2+(m-1)Px'+Q, =0$, and solve this quadratic to get the requisite value of $x'$.

To cause the fourth term to be absent it will be necessary to solve an equation of the third degree; the next coefficient would give an equation of the fourth degree, and so on upward. These equations would be difficult or impossible to solve.

EXAMPLES.

1. Transform $x^3-6x^2+7=0$ into an equation where the second term is absent. Ans. $y^3-12y-9=0$.

2. Transform $x^4-8x^3-5x+12=0$ into an equation whose second term is wanting.

Ans. $y^4-24y^2-69y-46=0$.

It sometimes happens that the same value of $x'$ will satisfy both the equations arising from putting the coefficients of the second and third terms $=0$. In this case those two terms will vanish simultaneously.

As an example: $x^4+4x^3+6x^2+3x+4=0$.

Here $\frac{-P}{m}=\frac{-4}{4}=-1=x'$, and substituting $y+x'=y-1$ for $x$:

$$\begin{array}{rrrrrl}
y^4 & -4y^3 & +6y^2 & -4y & +1 & \\
 & 4y^3 & -12y^2 & +12y & -4 & \\
 & & 6y^2 & -12y & +6 & \\
 & & & 3y & -3 & \\
 & & & & +4 & \\
\hline
y^4 & & & -y & +4 & =0
\end{array}$$

is the transformed equation.

Or by successive divisions:

$$
\begin{array}{lllll}
x^4+4x^3+6x^2+3x+4 \mid x+1 & & & & \\
x^4+\ x^3 & x^3+3x^2+3x \mid x+1 & & & \\
\quad 3x^3+6x^2 & x^3+\ x^2 & x^2+2x+1 \mid x+1 & & \\
\quad 3x^3+3x^2 & \quad 2x^2+3x & x^2+\ x & x+1 \mid x+1 & \\
\qquad 3x^2+3x & \quad 2x^2+2x & \qquad x+1 & x+1 & 1 \\
\qquad 3x^2+3x & \qquad\quad x & \qquad x+1 & \quad ,0 & \\
\qquad\qquad ,+4 & \qquad x+1 & \qquad ,0 & & \\
 & \qquad ,-1 & & &
\end{array}
$$

Giving the remainders 4, —1, 0, 0, and consequently the equation $y^4-y+4=0$.

The same example by Synthetic Division is as follows:

$$
\begin{array}{l}
1\ +4\ +6\ +3\ +4 \mid -1 \\
\quad -1\ -3\ -3\ -0 \\
\hline
1\ +3\ +3\ +0,+4 \\
\quad -1\ -2\ -1 \\
\hline
1\ +2\ +1,-1 \\
\quad -1\ -1 \\
\hline
1\ +1,+0 \\
\quad -1 \\
\hline
1,+0
\end{array}
$$

Here the remainders are as before, and the equation is $y^4-y+4=0$.

### DERIVED POLYNOMIALS.

**Art. 50.** By examining equation (2) Art. 45, we see that the coefficient of $y^o$ is simply the first member of the equation which was transformed with a dash placed on $x$. Omit the dash, and let this be denoted by $f(x)$.

The coefficient of $y'$ is formed from $f(x)$ by multiplying each coefficient by the exponent of $x$ in the term and diminishing that exponent by unity. Let this coefficient be denoted by $f'(x)$.

The numerator of the coefficient of $y^2$ is formed from $f'(x)$ by the same law as that by which $f'(x)$ was derived from $f(x)$. The denominator is 2 or $1.2$. Let it be denoted by $\frac{f''(x)}{1.2}$.

And the law by which any of these coefficients is derived from its immediate predecessor is:

*Multiply each term of the preceding coefficient by the exponent of $x$ in that term, diminish this exponent by unity and divide the algebraic sum of the results by the number of preceding coefficients.*

These coefficients, after the first, are derived from their immediate predecessors by the same law as that by which the coefficients of the Binomial Formula are built up.

The student of Calculus will recognize the numerators as differential coefficients of the first, second, third order, etc. They are called derived functions, or *derived polynomials.*

Thus $f'(x)$ is the *first derived polynomial of $f(x)$.*

$f''(x)$ is the *second derived polynomial.*

$f'''(x)$ is the *third derived polynomial.*

etc., etc., etc.

Mark the distinction between the derived polynomials and the *coefficients of the development* in equation (2) Art. 45. The first coefficient is the original first member with $x'$ in place of $x$; the second *coefficient* is the *first derived polynomial;* the *third coefficient* is $\frac{1}{2}$ of the *second derived polynomial;* the *fourth coefficient,* $\frac{1}{6}$ or $\frac{1}{2.3}$ of the *third derived polynomial,* etc.

EXAMPLES.

Let $3x^4+6x^3-3x^2+2x+1=0$.

$f(x)=3x^4+6x^3-3x^2+2x+1$

$f'(x)=12x^3+18x^2-6x+2.$

$f''(x)=36x^2+36x-6.$

$f'''(x)=72x+36.$

$f''''(x)=72.$

The last terms having $x^o$, the terms into which 0 is multiplied do not appear in the succeeding derived polynomials. In this way 1, 2, —6, 36 and 72 are successi ely dropped, which terminates the series.

Let it be required to transform the equation

$$3x^3+15x^2+25x-3=0$$

into one wanting the second term. First placing it in the reduced form: $x^3+5x^2+\frac{25}{3}x-1=0$; whence clearing of fractions (Art. 37),

$$y^3+15y^2+75y-27=0.$$

$f(x)=y^3+15y^2+75y-27$ and $x'=-\frac{-P}{m}=-5$;

$$\therefore\ f(x)=-125+375-375-27=-152$$

$f'(x)=3y^2+30y+75$; $\qquad f'(x)=75-150+75=0$

$\frac{f''(x)}{2}=3y+15$; $\qquad \frac{f''(x)}{2}=-15+15=0$

$\frac{f'''(x)}{2.3}=1.$ $\qquad \frac{f'''(x)}{2.3}=1=1$

Hence the equation is $u^3-152=0$, an equation wanting both the second and third terms.

2. Transform $x^5-10x^4+7x^3+4x-9=0$ into an equation wanting the second term.

Ans. $u^5-33u^3-118u^2-152u-73=0$.

3. Transform $3x^4-13x^3+7x^2-8x-9=0$ into an equation the roots of which shall not be so great by the quantity $\frac{1}{3}$.

Ans. $3u^4-9u^3-4u^2-\frac{65}{9}u-\frac{34}{3}=0$.

RELATIONS OF THE DERIVED POLYNOMIALS TO THE ROOTS OF AN EQUATION—EQUAL ROOTS.

**Art. 51.** Let $a, b, c, d, \ldots l$ be the $m$ roots of

$$x'^m+Px'^{m-1}+Qx'^{m-2}+\ldots+Tx'+U=0.$$

Then we know that

$$x'^m + Px'^{m-1} + Qx'^{m-2} + \ldots + Tx' + U = (x'-a)(x'-b)(x'-c)$$
$$\ldots\ldots\ldots(x'-l).$$

Let this be transformed by substituting $u+x$ for $x'$:

$$(u+x)^m + P(u+x)^{m-1} + \ldots\ldots = (u+x-a)(u+x-b)(u+x-c)$$
$$\ldots(u+x-l) = [u+(x-a)][u+(x-b)][u+(x-c)]\ldots$$
$$[u+(x-l)]\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now, if we regard $(x-a)$, $(x-b)$, $(x-c)$, etc., as single quantities, the factors of the second or third member of equation (1) will be in the form of the continued product of the binomial factors of the first degree which belong to the roots of an equation. In this equation the unknown quantity is $u$.

If the indicated operations be performed in the first and last members of equation (1), we shall have from the first member:

$$f(x) + f'(x)u + \frac{f''(x)}{2}u^2 + \ldots\ldots u^m;$$

$f(x)$ being the first member of the equation with which we started, but without putting on the dash ($'$), and $f'(x)$, $f''(x)$, etc., being the derived polynomials. This expression placed opposite to what the second member becomes is an identical equation, to which the principle of Indeterminates' Coefficients applies. The coefficient of $u^o$ in this result will be the continued product of the second terms (to wit, $(x-a)$, $(x-b)$, $(x-c)$, etc., regarded as single terms) of the binomial factors of the first degree with respect to $u$ (Art. 20), and the coefficient of $u^o$ in the first member being $f(x)$,

$$f(x) = (x-a)(x-b)(x-c)\ldots(x-l),$$

which we already knew. But the coefficient of $u$ in the second member being the sum of the combinations of the $m$ second terms (or factors, $(x-a)$, $(x-b)$, etc.) in groups of $m-1$, we have:

$$f'(x) = \frac{f(x)}{x-a} + \frac{f(x)}{x-b} + \frac{f(x)}{x-c} + \ldots + \frac{f(x)}{x-l},$$

because, $f(x)$ being the product of the $m$ terms or factors, whenever it is divided by one of them, the quotient is product of a group of $m-1$.

Again, equating the coefficients of $u^2$, we get:

$$\frac{f''(x)}{2}=\frac{f(x)}{(x-a)(x-b)}+\frac{f(x)}{(x-a)(x-c)}+\ldots\ldots+\frac{f(x)}{(x-k)(x-l)}$$

and equating those of $u^3$:

$$\frac{f'''(x)}{2.3}=\frac{f(x)}{(x-a)(x-b)(x-c)}+\frac{f(x)}{(x-a)(x-b)(x-d)}+\ldots\ldots+\frac{f(x)}{(x-g)(x-k)(x-l)}$$

and so on.

Hence we may announce in general language that

*The* FIRST DERIVED POLYNOMIAL *of the first member of the reduced equation is equal to the algebraic sum of the quotients arising from dividing that first member successively and singly by the factors of the first degree belonging to the roots.*

*The* SECOND DERIVED POLYNOMIAL *is equal to twice the algebraic sum of the quotients arising from dividing the first member by the product of every group of two of the factors of the first degree belonging to the roots.*

*The* THIRD DERIVED POLYNOMIAL is six times the algebraic sum of similar quotients, but the divisors are the products of every group of three; and so on.

### EQUAL ROOTS.

**Art. 52.** The relations just discussed are very important, but specially interesting because they lead to the discovery of equal roots, if an equation has any.

Suppose an equation has two equal roots, $a$; then

$$f(x)=(x-a)(x-a)(x-b)\ldots.(x-l);$$

and the first derived polynomial,

$$f'(x)=\frac{f(x)}{x-a}+\frac{f(x)}{x-a}+\frac{f(x)}{x-b}+\text{etc.},$$

will have every term divisible by $x-a$, because the numerator of every term contains $(x-a)^2$, and after the denomin-

ator, even when it is $x-a$, is divided out, there is still a factor $x-a$ left. Thus we see there will be *a common divisor* of the first member and its first derived polynomial.

There might have been several roots of one value and several of another value. Thus, suppose there are $n$ roots $a$, $r$ roots $b$ and $s$ roots $c$; then

$$f(x) = (x-a)^n(x-b)^r(x-c)^s \dots (x-k)(x-l);$$

$$f'(x) = \frac{f(x)}{x-a} + \frac{f(x)}{x-a} + \dots\dots + \frac{f(x)}{x-b} + \frac{f(x)}{x-b} + \dots\dots + \frac{f(x)}{x-c} + \frac{f(x)}{x-c} + \dots\dots + \frac{f(x)}{x-k} + \frac{f(x)}{x-l}$$

Now, the terms of $f'(x)$ where the denominators are not repeated (like $x-k$ and $x-l$) will have as a factor $(x-a)^n (x-b)^r(x-c)^s$, and every other term will have a similar factor in which each repeated factor will have an exponent at least equal to $n-1$. Hence the first member, $f(x)$, and its first derived polynomial, $f'(x)$, will have a common divisor, which is $(x-a)^{n-1}(x-b)^{r-1}(x-c)^{s-1}$. And this will be the H.C.D., because none of the factors which belong to single roots can enter it, since every such factor would be wanting in some term of $f'(x)$ where it had been divided out; as, for instance, $x-k$ would be absent from $\frac{f(x)}{x-k}$.

**Art. 53.** *To find, then, the equal roots which may be in an equation, we find the H.C.D. between the first member and its first derived polynomial. If there is none, there are no equal roots; but if there is one, place it equal to* 0, *and the roots of this equation will be the* EQUAL *roots of the proposed equation.*

Call the H.C.D., $D$. If $D$ is of the first degree, there are two roots which are the same.

If $D$ is of the second degree and of the form $(x-a)^2$, there are *three roots a;* and if it is of the form $(x-a)(x-b)$, there are two roots $a$ and two roots $b$.

In general, whatever the degree of the equation $D=0$, each of its single roots will be twice a root of the equation proposed, and all of its repeated roots will appear once more frequently in the proposed equation.

Having found all the equal roots, make a continued product of the binomial factors corresponding to these, and divide $f(x)$ by it; this will lower or depress the degree of the equation as many units as there are equal roots and render it far easier to be solved, and may even bring the depressed equation within the limits of those which we know how to solve directly and exactly.

EXAMPLES.

1. Find the equal roots in the equation

$$x^7-3x^6+9x^5-19x^4+27x^3-33x^2+27x-9=0.$$

The first derived polynomial is:

$$7x^6-18x^5+45x^4-76x^3+81x^2-66x+27.$$

And the H.C.D. between this and the first member is:

$$x^4-2x^3+4x^2-6x+3.$$

Placing this equal to zero, and finding the H.C.D. between this first member *and its first derived polynomial*, we get $x-1$. Then $(x-1)^2$ is a factor of the first member of the secondary equation and $(x-1)^3$ is a factor similarly in the first member of the original equation. There are now known to be three roots $=1$. Dividing $(x-1)^2$ out of $x^4-2x^3+4x^2-6x+3=0$, we have $x^2+3=0$, and $x=\pm\sqrt{-3}$. $(x^2+3)^2$ will be a factor of the original first member and the product of the factors corresponding to the equal roots of the proposed equation is $(x-1)^3(x^2+3)^2=$ original first member.

2. What are the equal roots in

$$2x^4-12x^3+19x^2-6x+9=0?$$

The first derived polynomial is $8x^3-36x^2+38x-6$, and the H.C.D. $=x-3$.

There are two roots $= 3$, and the others, after dividing out and depressing to an equation of the second degree, are found to be $\frac{\sqrt{-2}}{2}$ and $\frac{-\sqrt{-2}}{2}$.

3. Find the equal and other roots of the equation $x^6 + 2x^5 - 12x^4 - 14x^3 + 47x^2 + 12x - 36 = 0$.

Ans. two $= 2$, two $= -3$ and also 1 and $-1$.

4. What are the roots of $x^5 + 4x^4 - 14x^2 - 17x - 6 = 0$?

Ans. three $= -1$, and besides 2 and $-3$.

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## CHAPTER VI.

### Limits and Places of Roots.

**Art. 53.** *A rational integral function of $x$ is one in which the exponents of $x$ are whole numbers and the coefficients are independent of $x$.*

Thus $f(x) = x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx^2 + Vx + U$, in which $m$ is a positive whole number (integer) and $P$, $Q$, $T$, etc., are independent of $x$, is a rational integral function of $x$ of the $m$th degree.

**Art. 54.** *In any rational integral function of $x$ arranged according to the descending powers of $x$, any term which is present may be made to contain the sum of all which follow it, as many times as we please, by taking $x$ large enough.*

*And any such term may be made to contain the sum of all which precede it by taking $x$ small enough.*

In $f(x) = x^m + Px^{m-1} + Qx^{m-2} + \ldots + Sx^{m-n+1} + Tx^{m-n} + \ldots + U$, $Sx^{m-n+1}$ will be the $n$th term, and may contain the sum of all which follow it, if $x$ be large enough. If it can be made to contain something larger than that sum, it will, of course, contain that sum. Now, suppose all the

terms after $Sx^{m-n+1}$ to have the largest coefficient among them. Let it be $L$; then $L(x^{m-n}+x^{m-n-1}+x^{m-n-2}+\ldots+x+1) >$ the sum of the terms following the $n$th, and $=$ $L\left(\frac{x^{m-n+1}-1}{x-1}\right)$. Divide the $n$th term by this: $\frac{Sx^{m-n+1}}{\frac{L(x^{m-n+1}-1)}{x-1}}$

$$=\frac{S}{L}\times\frac{x^{m-n+1}(x-1)}{x^{m-n+1}-1}=\frac{S}{L}\times\frac{x-1}{1-\frac{1}{x^{m-n+1}}}.$$

By increasing $x$ we may increase the numerator indefinitely, and at the same time make the denominator as near unity as we please. Consequently the $n$th term will contain those that follow as many times as desired. This proves the first part of the proposition.

Suppose we make $x=\frac{1}{y}$; then increasing $y$ diminishes $x$.

We have: $\frac{1}{y^m}[1+Py+Qy^2+\ldots+Sy^{n-1}+Ty^n+\ldots+Uy^m]$. The series within the brackets is such that any term, as $Sy^{n-1}$, may be made to contain the sum of all which precede it, $1+Py+Qy^2+$ etc., as often as we please, by taking $y$ large enough, which means $x$ small enough. This is evidently shown by the same reasoning as in the first case, and establishes the second branch of the proposition.

**Art. 55.** The first term of the function may be made to contain the sum of all of its successors any number of times.

**Art. 56.** A variable quantity is said to increase or decrease under the *law of continuity*, when, in passing from one designated state to another, it passes through every intermediate state without interruption. A taper burning away, a cask of fluid being discharged by a cock, a plant growing, present instances.

Let it be shown that if $x$ increases or decreases under the law of continuity, that $f(x)$ will increase or decrease under the same law.

When $x = a$, let $f(a)$ designate the corresponding state of the function; when $x = b$, $f(b)$ the state of the function then corresponding, etc. Suppose that $x'$ were a certain value of $x$, and give it a small increment, $u$. We see from the development (2) of Art. 45, that we shall have:

$$f(x'+u) = f(x') + uf'(x') + \frac{u^2}{1.2}f''(x') + \frac{u^3}{1.2.3}f'''(x') + \ldots\ldots + \frac{u^m}{1.2.3\ldots m}f^{m\prime}(x').$$

In (2) of Art. 45, $\frac{u^m}{1.2.3\ldots m}f^{m\prime}(x') \doteq u^m$.

Now, if we transpose the term $f(x')$ to the first member, we have:

$$f(x'+u) - f(x') = uf'(x') + \frac{u^2}{1.2}f''(x') + \ldots\ldots\ldots\ldots + \frac{u^m}{1.2.3\ldots m}f^{m\prime}(x').$$

The first term in the second member (which is present) may be made indefinitely greater than the sum of all which here follow it (they would precede it in the arrangement of Art. 55), by taking the increment, $u$, small enough. But when $u$ is taken extremely minute, although the first term in the second member will contain the sum of the following terms an indefinite number of times, the first term itself becomes indefinitely small. Hence the difference between the states of the function, $f(x'+u) - f(x')$, becomes *inappreciable*. Hence, when the successive increments of $x$ are indefinitely small, and $x$ varies under the law of continuity, the function of $x$ will vary under the same law.

### LIMITS OF ROOTS.

**Art. 57.** The limits of the roots of an equation are values between which all the roots exist.

*A superior limit of the positive roots is any number or quantity of their kind greater than the greatest of them.*

*An inferior limit of the positive roots is any number or quantity of their kind less than the least of them.*

*A superior limit of the negative roots is a number or quantity of their kind which is negative but numerically greater than all of the negative roots.*

*An inferior limit of the negative roots is a number or quantity of their kind which is negative but numerically less than any of the negative roots.*

Since a root of an equation whose second member is zero, when substituted for the unknown quantity, will reduce the first member to zero, if we put for the unknown quantity any quantity greater than the greatest of the positive roots, the first member, when reduced, will be found greater than zero, that is, positive. This number and all greater than it, that is, all between it and $+\infty$, would be *superior limits of the positive roots* of the equation. The smallest of such limits which is attainable is, of course, the one required for practical use, as a general thing.

It may be proved that *the greatest coefficient plus* 1 is a superior limit of the positive roots; and even *the greatest negative coefficient plus* 1 is such a limit, and often a better one, because it may be smaller. This last is called

MACLAURIN'S LIMIT.

In the equation $f(x)=x^m+Px^{m-1}+Qx^{m-2}+\ldots.+Tx+U=0$, ....(1), let the first term be positive and the others either positive or negative as may happen. Let $N$ be the greatest negative coefficient, and suppose what would be the most unfavorable case which could happen, that all the other coefficients except the first were equal to it, *and all negative.* These negative terms would then form a geometrical progression with the ratio $x$, and it would be necessary only to put for $x$ a value which would make

$$x^m>N(x^{m-1}+x^{m-2}+x^{m-3}+\ldots.+x+1)=\frac{N(x^m-1)}{x-1}.$$

Now, if in the inequation $x^m<\dfrac{N(x^m-1)}{x-1}$ we place $x-1$ equal

to $N$, or $x = N+1$, we shall satisfy it, having $x^m > x^m - 1$.

*Hence the greatest negative coefficient plus* 1 *is a superior limit of the positive roots of an equation.*

**Art. 58.** Since changing the signs of the alternate terms would make the positive roots all negative (and the negative all positive) it is evident that *the greatest negative coefficient of the transformed equation plus unity would be a superior limit of the negative roots of the equation;* that is, intrinsically less than all of the roots. It would be numerically greater than any negative root.

ORDINARY SUPERIOR LIMIT OF POSITIVE ROOTS.

**Art. 59.** When the first term is followed immediately by one or more other positive terms, a closer limit may be obtained.

Let us suppose $x^{m-n}$ to be the power of $x$ in the first negative term, and take the most unfavorable case which could happen, that is, that all the succeeding terms are negative and all have the greatest coefficient among them. Let $S$ be that coefficient. Then if we can make $x^m > Sx^{m-n} + Sx^{m-n-1} + \ldots + Sx + S$, it will be more than sufficient to make the first member positive, because, in fact, $x^m$ would be increased by the addition of the other positive terms.

Divide both members of the inequation by $x^m$, and we get:

$$1 < \frac{S}{x^n} + \frac{S}{x^{n+1}} + \frac{S}{x^{n+2}} + \ldots\ldots + \frac{S}{x^{m-1}} + \frac{S}{x^m}.$$

There are $n$ terms before the first negative, term and let us suppose $x = \sqrt[n]{S} + 1$; representing the value of $\sqrt[n]{S}$ by $S'$, whence $S = S'^n$, and $x = 1 + S'$, the second member of the inequality will become:

$$\frac{S'^n}{(S'+1)^n} + \frac{S'^n}{(S'+1)^{n+1}} + \ldots + \frac{S'^n}{(S'+1)^{m-1}} + \frac{S'^n}{(S'+1)^m} < 1.$$

We have here a geometrical progression, in which the first term is $\frac{S'^n}{(S'+1)^n}$, with a ratio $\frac{1}{(S'+1)}$. Its sum, therefore, is:

$$\frac{\frac{S'^n}{(S'+1)^{m+1}} - \frac{S'^n}{(S'+1)}}{\frac{1}{S'+1} - 1} = \frac{\frac{S'^n}{(S'+1)^{m+1}} - \frac{S'^n}{(S'+1)^n}}{\frac{-S'}{S'+1}} =$$

$$\frac{S'^{n-1}}{(S'+1)^{n-1}} - \frac{S'^{n-1}}{(S'+1)^m},$$

which is the difference between two proper fractions, and therefore less than 1, as was required. The quantity $\sqrt[n]{S}+1$ will consequently make the first member of the given equation positive, and be a superior limit of the positive roots. This result may be stated in common language thus:

*Extract that root of the greatest negative coefficient of which the index is the number of terms before the first negative term; increase this by* 1, *and the result will be a good superior limit of the positive roots of the equation.* If any term is absent, it must be counted to determine the index of the root.

If $n = 1$, the second term is negative, and $\sqrt[1]{S}+1 = S+1$, the same as in Art. 57.

EXAMPLES.

Find a superior limit of the positive roots of $x^4+11x^2-25x-67=0$.

Here $n=3$, and the limit is $\sqrt[3]{67}+1$. The cube root of 67 is between 4 and 5, and hence $5+1$ will be the limit.

2. $x^4+11x^2-25x-61=0$. Limit $=\sqrt[3]{61}+1$, or 5.

INFERIOR LIMIT OF POSITIVE ROOTS.

**Art. 60.** If in any equation we make $x=\frac{1}{y}$, the roots

of the transformed equation being reciprocals of those in the first, *the greatest positive* root of the transformed will be the *reciprocal of the least positive* root of the given equation. Hence to obtain the inferior limit of the positive roots: Substitute $\frac{1}{y}$ for $x$; find the superior limit of the positive roots of the transformed equation; its reciprocal will be the limit required.

### SUPERIOR LIMIT OF NEGATIVE ROOTS.

**Art. 61.** This, as already indicated (Art. 58), will be the superior limit of the positive roots of an equation whose roots have signs opposite to those of the given equation. This transformed equation can be had by making $x = -y$, or by changing the signs of the alternate terms. This limit is *numerically* superior, but not algebraically or in fact.

### INFERIOR LIMIT OF NEGATIVE ROOTS.

**Art. 62.** Take the reciprocal of the last transformed equation, that is, put $x = -\frac{1}{y}$, and find a superior limit of the positive roots of this equation, it will be the required limit, because, since $x = -\frac{1}{y}$, we have $y = -\frac{1}{x}$, and *the greatest positive value of y will correspond to the least (numerically considered) negative value of x.*

### NEWTON'S LIMIT.

**Art. 63.** Any number, which on being substituted for the unknown quantity in the first member of an equation and in its derived polynomials, makes them all positive, is a superior limit of the positive roots.

If the roots $a, b, c, \ldots . l$ of $f(x) = 0$ be diminished by $x'$, that is, if we make $x = x' + y$, we shall have, eq. (2), Art. 45:

$$f(x')+f'(x')y+f''(x')\frac{y^2}{1.2}+\ldots+f^{(m-1)'}(x')\frac{y^{m-1}}{1.2\ldots(n-1)}+y^m = 0.$$

If such a value be placed in this equation for $x'$ as to make all the terms positive, we know that all its roots, that is, values of $y$, must be negative, and from the relation $x=x'+y$, we have $y=x-x'$, so that $y$ being negative, $x'>x$, and consequently, whatever value will make the first derived polynomial, $f'(x')$, positive will make positive the original first member, where the coefficients are the same, but $x$ takes the place of a quantity greater by $x'$.

### EXAMPLE.

Find a superior limit to the positive roots of $x^3-5x^2+7x-1=0$.

We need not retain the dashes upon the $x$, but write:

$$f(x)=x^3-5x^2+7x-1.$$
$$f'(x)=3x^2-10x+7.$$
$$\frac{f''(x)}{1.2}=3x-5.$$
$$\frac{f'''(x)}{1.2.3}=1.$$

Beginning at the last derived function in which $x$ appears, and substituting the smallest whole number which will make it positive, we see that 3 makes it positive. Likewise the next before it, and so on to the last. 3, then, is the limit.

### EXAMPLE 2.

What is the superior limit of the roots of $x^5-5x^4-13x^3+17x^2-69=0$?

We have *derived polynomials* as follow (after dividing out their appropriate denominators):

$$5x^4-20x^3-39x^2+34x.$$
$$10x^3-30x^2-39x+17.$$
$$10x^2-20x-13.$$
$$5x-4.$$
$$1.$$

1 placed for $x$ gives $5 - - 4 = 1$, positive; but fails in the next above. 2 fails, but 3 gives a positive result. 3, when tried in $f''(x)$, fails, and so does 4, by a single unit, 5, being tried, gives $+$, and being tried in $f'(x)$, fails, and so does 6. And 7 is found to be the required limit.

It will be perceived that this has given us the smallest limit in whole numbers, and it will always give us a closer limit than any of the previous methods. The amount of computation confines its use to cases where closeness of limit is important. It was invented by the immortal Newton, who has shed brilliant and enduring light upon all of the many branches of learning to which he addressed himself.

### BUDAN'S TEST OF IMAGINARY ROOTS.

**Art. 64.** If the roots of an equation be reduced by a quantity $r$, and the transformed equation shows a *loss* of $m$ variations of signs, and if the reciprocal equation be reduced by $\frac{1}{r}$, and this transformed equation shows $n$ variations, which were not lost but which remain; then there are $m - n$ imaginary roots between $r$ and 0.

Because, in reducing the roots of the equation by $r$, all positive roots less than $r$ will have become negative*, and there will be as many positive roots between 0 and $r$ as there have been positive roots changed into negative, which is to say, as many as there have been variations lost, whereas, in reducing the roots of the reciprocal equation by $\frac{1}{r}$, no positive root greater than $\frac{1}{r}$ will be changed. But should a different result appear, it would indicate the existence of imaginary roots, the number of which within these limits will be the number of variations lost by the

* The factors of the first degree belonging to *negative roots* are of the form $x+c$, $x+d$, etc., and in the multiplication which builds up the first member of the reduced equation, they exercise no influence on the signs or number of variations.

first transformation minus the number not lost in the one last described.

Now suppose $r$ was a superior limit of the positive roots; when we reduced by $r$, the number of lost variations would be equal to the number of positive roots, provided they were all real. And in the reciprocal equation $\frac{1}{r}$ would be an inferior limit of the positive roots, and when it was transformed by reducing the roots by the quantity $\frac{1}{r}$, the transformed equation would show no loss of variations, provided the positive roots were all real. A different result would show that there was an absurdity or contradiction about some of the roots, which we would therefore perceive to be imaginary. And these would appear to be positive.

And the number of imaginary roots thus discovered would be the number of variations lost in the transformation of the original equation *minus the number not lost* or which remain in the transformation from the reciprocal equation.

Again take the original equation and change the alternate signs; the positive roots will be turned into negative, and the negative into positive roots. Proceed with this as with the original equation, and we shall discover the number of imaginary roots, apparently negative.

EXAMPLE.

Find the number of imaginary roots in

$$x^5+3x^4+2x^3-3x^2-2x-2=0.$$

Since this equation is of an odd degree, with the absolute term negative, there is at least one real root positive, and since there is but one variation, there is but one such root. We need not look among the positive roots for imaginary roots, but according to Budan's Test, we change

the alternate signs and have $x^5-3x^4+2x^3+3x^2-2x+2$, of which 1 is a superior limit.

| COEFFICIENTS OF DIRECT EQUATION. | COEFFICIENTS OF RECIPROCAL EQUATION. |
|---|---|
| $1 \ -3 \ +2 \ +3 \ -2 \ +2 \ \vert \ +1$ | $2 \ -2 \ +3 \ +2 \ -3 \ +1 \ \vert \ +1$ |
| $+1 \ -2 \ \pm 0 \ +3 \ +1$ | $+2 \ \pm 0 \ +3 \ +5 \ +2$ |
| $-2 \ \pm 0 \ +3 \ +1, +3$ | $\pm 0 \ +3 \ +5 \ +2, +3$ |
| $+1 \ -1 \ -1 \ +2$ | $+2 \ +2 \ +5 \ +10$ |
| $-1 \ -1 \ +2, +3$ | $+2 \ +5 +10, +12$ |
| $+1 \ \pm 0 \ -1$ | $+2 \ +4 \ +9$ |
| $\pm 0 \ -1, +1$ | $+4 \ +9, +19$ |
| $+1 \ +1$ | $+2 \ +6$ |
| $+1, \pm 0$ | $+6, +15$ |
| $+1$ | $+2$ |
| $+2$ | $+8$ |
| $1 \ +2 \ \pm 0 \ +1 \ +3 \ +3$ | $2 \ +8 +15 +19 +12 +3$ |

Since in the coefficients of the transformation from the direct, the third, $\pm 0$, is between two terms of like signs, we know from De Gua's Test, that there are two imaginary roots in the transformed equation; we may therefore use the plus sign, which shows 4 variations lost. In the transformation of the reciprocal equation there are none left; hence $4-0=4$, the number of imaginary roots.

2. In $x^5-10x^3+6x+1=0$, how many imaginary roots?

Ans. All real.

3. How many imaginary roots in $x^4-4x^3+8x^2-16x+20=0$?

Ans. None.

4. In $x^4+x^3+x^2+3x-100=0$, how many imaginary roots?

Ans. 2 imaginary roots and 2 real with opposite signs.

### PLACES OF REAL ROOTS.

**Art. 65.** It has been shown that if $x$ varies under the law of continuity that $f(x)=x^m+Px^{m-1}+\ldots+Tx+U$ will do

so likewise. Let us suppose that we had substituted for $x$ in $f(x)$ a number, $p$, and the result was greater than 0, or $+$. Then, if $x$ decreases under continuity, it will after a time, come upon the value of one of the roots, when the result will be 0. Continuing to decrease, its value (say $q$) will give a result less than 0 or $-$, and consequently we say that *if two numbers, $p$ and $q$, when substituted for the unknown quantity in the first member of an equation of which the second member is 0, give results with opposite signs, there is at least one real root between $p$ and $q$.*

A quantity may change its sign by passing through infinity as well as through 0. Let $x = \frac{1}{y}$; here as $y$ decreases, $x$ increases; when $y$ is very small, $x$ becomes very great; when $y = 0$, $x = \infty$; when $y < 0$, or negative, $x$ becomes negative; but in the rational integral function, which is the first member of the equation, no finite value of $x$, as between $p$ and $q$, could make $f(x) = \infty$.

There might be more than *one* root between $p$ and $q$. Moreover, if there are roots between $p$ and $q$, the substitution of $p$ and $q$ will not necessarily produce results with contrary signs, for,

**Art. 66.** *When an odd number of roots lie between $p$ and $q$, their substitution will give results having opposite signs; when an even number of roots lie between them the results will have the same signs.*

Suppose that there were several roots, $a$, $b$, $c$, etc., between $p$ and $q$, and some others besides. Let the product of the factors of the first degree with respect to these latter roots be $Y$; then we shall have:

$$f(x) = (x-a)(x-b)(x-c) \ldots\ldots \times Y = 0.$$

Substitute for $x$ first $p$ and then $q$; let $Y'$ be what $Y$ becomes on the substitution of $p$, $Y''$ the result of substituting $q$ in $Y$. Now $Y'$ and $Y''$ will have the same sign; otherwise, by Art. 65, there would be another root, or

roots, lying between $p$ and $q$, which is contrary to the supposition.

If we now make the substitutions, and for convenience write the first result over the other, we shall have:

$$\frac{(p-a)(p-b)(p-c)\ldots\ldots\times Y'}{(q-a)(q-b)(q-c)\ldots\ldots\times Y''}$$

or otherwise thus:

$$\frac{p-a}{q-a}\times\frac{p-b}{q-b}\times\frac{p-c}{q-c}\times\ldots\ldots\ldots\times\frac{Y'}{Y''}.$$

Suppose $p>q$, then $a$, $b$, $c$, etc., will be $<p$ and $>q$, so that all the quotients will be negative except $\frac{Y'}{Y''}$.

Now, if the number of roots $a$, $b$, $c$, etc., between $p$ and $q$ is even, the product of these fractions will be positive, and the first result divided by the second will have a positive quotient, that is, the results of the substitutions of $p$ and $q$ will have the same sign, and the contrary will be true when $a$, $b$, $c$, etc., are odd in number.

## THE THEOREM OF STURM.

**Art. 67.** But the best of all the modes yet discovered of determining the character and places of the roots of an equation is the celebrated theorem of Sturm, contributed in 1829 to the scientific world by that eminent French mathematician. The object of *Sturm's theorem is to discover the number of real and imaginary roots in any equation, and the places of the real roots.*

Sturm's Theorem does all that is accomplished by the methods which have thus far been examined, and more beside. Still those methods should be preserved, because they are sometimes sufficient for the purpose in hand and of easier application than the theorem of Sturm.

This theorem deals with the signs of certain functions of the unknown quantity, which are: the first member of the

equation, its first derived polynomial and certain others which are formed in the following manner: First free the equation of equal roots, if it has any; then apply to $f(x)$, the first member, and to $f'(x)$, its first derived polynomial, the process for finding the H.C.D., but with this difference—after each remainder has been found, *change its sign*, and during the intermediate operations neither introduce nor suppress any factor but *a positive one*.

Let $f(x) = Nx^m + Px^{m-1} + Qx^{m-2} + \ldots\ldots + Tx + U = 0$ be the equation, and designate $f(x)$ by $V$, and $f'(x)$ by $V_1$, and by $-V_2$, $-V_3$, $-V_4$, $-V_r$, the remainders of the various divisions wherein the quotients were: $Q_1$, $Q_2$, $Q_3$, .... $Q_{r-1}$.

We shall have the following equations:

$$
\begin{aligned}
V &= V_1Q_1 - V_2 \\
V_1 &= V_2Q_2 - V_3 \\
V_2 &= V_3Q_3 - V_4 \\
&\ldots\ldots\ldots \\
&\ldots\ldots\ldots \\
V_{r-2} &= V_{r-1}Q_{r-1} - V_r \ldots\ldots (1)
\end{aligned}
$$

$V_r$ cannot be 0, otherwise there would be a C.D. between $f(x)$ and $f'(x)$, which is contrary to the supposition. It must, then, be a number, because the operation is to be carried on until the last remainder is independent of $x$.

**Art. 68.** Let $A$ and $B$ be two numbers, and $A < B$. Let $A$ be substituted for $x$ in the expressions, $V$, $V_1$, $V_2$, $V_3$, etc., and the signs of the results recorded; then substitute $B$ and record the signs. The sign of $V_r$ will always remain the same, being independent of $x$. Then the theorem declares that

*The number of variations in the first series of signs, diminished by the number of variations in the second, will be equal to the number of real roots between A and B.*

**Art. 69.** To show this, it will be convenient first to establish three lemmas, as follow:

FIRST.

*No two consecutive functions, $V$, $V_1$, $V_2$, etc., can become 0 for the same value attributed to $x$.*

Let us take any equation out of the group (1), as

$$V_{n-1} = V_n Q_n, - V_{n+1},$$

and suppose that $V_{n-1}$ and $V_n$ should both vanish for a value of $x$, then from the equation, $V_{n+1}$, would also be zero. And the next equation of the series, having $V_n$ and $V_{n+1}$, both 0, would give $V_{n+2} = 0$, and so on. Thus they would all vanish and the last equation would give $V_r = 0$, which cannot be.

SECOND.

**Art. 70.** *When any one of these functions becomes 0, the one before it will have a different sign from the one following it* for the same value of $x$, as is shown by taking any one of the equations, as $V_2 = V_3 Q_3 - V_4$ and letting $V_3 = 0$ $\therefore V_2 = - V_4$.

THIRD.

**Art. 71.** *If a number almost equal to one of the real roots of the equation be substituted for the unknown quantity in the first member, and likewise in its first derived polynomial, the results will have contrary signs; but if the substituted quantity be greater than this root by an extremely small amount the signs of the results will be the same.*

Let us suppose that $a$ were a root and the added small quantity $u$. Let $a+u$ and $a-u$ be substituted for $x$; then by (2) Art. 45 we shall have:

$$f(a+u) = f(a) + f'(a)u + f''(a)\frac{u^2}{1.2} + \ldots\ldots$$

$$\ldots . + f^{(m-1)\prime}(a)\frac{u^{m-1}}{1.2.3..(m-1)} + u^m \ldots .(1)$$

$$f(a-u) = f(a) - f'(a)u + f''(a)\frac{u^2}{1.2} - \ldots$$

$$\pm f^{(m+1)'}(a)\frac{u^{m-1}}{1.2.3..(m-1)} \mp u^m \ldots (2)$$

In these $f(a) = 0$, and as $a$ is a very minute quantity, the first terms of those portions of the series which remain will far exceed in value the sum of the remaining terms (Art. 54) and will give sign to the series. These series will then become:

$$u[f'(a) + \frac{u}{1.2}f''(a) + \ldots + u^{m-1}] \ldots (3)$$

and $$-u[f'(a) - \frac{u}{1.2}f''(a) + \ldots \mp u^{m-1}] \ldots (4)$$

The upper one will be positive and the lower will be negative, and $f'(a)$ positive all the time, as they are here shown. In all cases, it is apparent that the sign of the lower series will be different from $f'(a)$, and that of the upper the same. But (4) and (2), which are the same, are the result of substituting $a-u$ in $V$, while (3) and (1) are the result of substituting $a+u$. $f'(a)$ is the result of substituting $a$ for $x$ in $V_1$. The truth of the lemma is, then, demonstrated.

**Art. 72.** Then to demonstrate the theorem of Sturm, let us suppose a varying quantity, $A$, which at the outset is less than the least of the real roots of $V = 0$, $V_1 = 0$, $V_2 = 0, \ldots V_n = 0, \ldots$ and $V_{r-1} = 0$, that is, of all the equations formed by putting Sturm's functions equal to 0. Let it be substituted for $x$ in all of them, and record the signs of the results. Afterwards suppose $A$ to grow: after a time it will be equal to the least of the roots mentioned above, and some *one* of the functions $V_1$, $V_2$, etc., will vanish. But, as its sign agrees with the one before it and disagrees with the one after it (Art. 70), or else agrees with the one behind it and disagrees with the one before it, *the number of variations will not be affected.*

And this will be true even if two, or several, of the func-

tions vanish at the same time; because the same conditions as those just described would hold when the vanishing functions were separated from each other by intervals in the series of them. And this must be the case because no two consecutive ones can vanish simultaneously (Art. 69). This will continue until the varying value of $A$ arrives in the close neighborhood of a root of $V = 0$, that is, the original equation, when the signs of $V$ and $V_1$ will be different, giving a variation, and after it passes the value of the root, and $-u$ becomes $+u$, they will agree in sign, giving a permanence, *or losing one variation* (Art. 71).

If it be supposed that $A$ continues to grow until it arrives in close proximity to another real root of $T = 0$, the same thing will take place, and when $V$ becomes zero and emerges with a change of sign, *it will have passed another root, and another variation will have been lost.*

And so every time a real root is passed, a variation of the signs of the functions $V$, $V_1$, $V_2$, etc., will have been lost, until $A$ has grown greater than the greatest real root of the original equation. The number of variations lost will be equal to the number of real roots between $A$ and $B$, and as they were taken as the numerically superior limit of the negative roots and the superior limit of the positive roots, *the total number of real roots will be known.* This number subtracted from the exponent which shows the degree of the equation will give *the number of imaginary roots.* This last must of course be 0 or an even number, and thus the theorem is found to be true.

**Art. 73.** If in finding $V_1$, $V_2$, etc., any one of the functions placed $= 0$ should give an equation all of the roots of which were imaginary (and this fact would be known by the function remaining of one sign for all real values of $x$,* the work need proceed no farther.

The polynomial function remaining always remaining of

* NOTE.—When the first member is constantly of the same sign for all real values of $x$, we infer that all the roots are imaginary, because if one value resulted in plus and another in minus, there would be a real root between the numbers substituted.

the same sign and the last one, $V_r$, being constant, none of those between them can change sign. And therefore, if any loss of variations takes place, it must do so among those which precede. This conclusion will be reached by considering the chain of equations which connect the polynomial spoken of with $V_r$.

Both ends of the chain remaining of a constant sign, the nature of the connection is such as to prevent all intermediate functions from changing. Or otherwise thus: Take the equations

$$V_{r-4} = V_{r-3}\ Q_{r-3} - V_{r-2} \quad \ldots\ldots (1)$$
$$V_{r-3} = V_{r-2}\ Q_{r-2} - V_{r-1} \quad \ldots\ldots (2)$$
$$V_{r-2} = V_{r-1}\ Q_{r-1} - V_{r} \quad \ldots\ldots (3)$$

Suppose that it was found that $V_{r-2}$ would not change sign. Since the quotients $Q_1$, $Q_2$ .... $Q_{r-1}$ would have no influence on the sign, as the values of $x$ are not substituted in them, if we transpose $V_r$ to the first member then that member being fixed in sign, $V_{r-1}$, would be so likewise.

If $V_{r-3}$ were found constant in sign we might eliminate $V_{r-1}$ out of (2) by substituting its value (3) and a single equation would result with $V_{r-2}$ and quantities fixed in sign; hence $V_{r-2}$ could not change sign and in the same way it could be shown that $V_{r-1}$ could not change sign.

If $V_{r-4}$ were found to be constant in sign, $V_{r-3}$ and $V_{r-2}$ could be eliminated by substituting the values from (2) and (3) whence it would be seen that $V_{r-1}$ must not change sign; then in succession the other two. And so for any function.

**Art. 74.** After the varying value of $x$ has passed above and below the greatest and least roots, the further increase or decrease can make no difference in the signs; and this is true up to any extent even to $+\infty$ and $-\infty$. The substitution of $+\infty$ and $-\infty$ will cause the functions to take their signs from their first terms, and will be found convenient because we need substitute in the first term only.

This will give the *number* of the real roots. But in ad-

dition to this we may wish to know *their places*, that is, between what whole numbers they may lie. To do this we substitute for $A$ and $B$, 0 and 1, 1 and 2, 2 and 3, 3 and 4, etc. Thus having found one or more roots between 2 and 3 we know that the roots are 2 + a fraction less than unity.

For the negative roots we substitute — 1 and 0, — 2 and — 1, — 3 and — 2, and so on. Having found a root between — 3 and —2 it will be — 3 plus a fraction, etc. Its initial figure will be —2, as —2.57.

EXAMPLE.

1. $8x^3-6x-1=0$. $f'(x)=8x^3-6x-1$

$f'(x)=24x^2-6$. Suppress the positive factor 6 in $f'(x)$ and we have $V_1=4x^2-1$.

$$\begin{array}{l|l} 8x^3-6x-1 & 4x^2-1 \\ \underline{8x^3-2x} & 2x \\ \quad -4x-1. & \end{array}$$

Here changing sign we have $4x+1=V_2$. Multiplying $V_1$ by the positive factor 4 and we get $16x^2-4$ to be divided by $4x+1$.

$$\begin{array}{l|l} 16x^2-4 & 4x+1 \\ \underline{16x^2+4x} & 4x-1 \\ \quad -4x-4 & \\ \quad \underline{-4x-1} & \\ \qquad -3 & \end{array}$$

—3 and + 3 and + 3 = $V_r$. Hence the polynomials are

| | minus $\infty$. | | plus $\infty$. | |
|---|---|---|---|---|
| $V=8x^3-6x-1$ | — | 3 variations. | + | 0 variation |
| $V_1=4x^2-1$ | + | | + | |
| $V_2=4x+1$ | — | | + | |
| $V_r=\ +3$ | + | | + | |

Hence 3 variations have been lost and *all* the roots are real. To determine their places: substitute 0 and 1, 0 and — 1.

| 0 and 1 | |
|---|---|
| — | + |
| — | + |
| + | + |
| + | + |

One variation lost and hence there lies between 1 and 0 one real root, which is zero plus a fraction. There are no more positive real roots because + 1 gives the

same signs as $+\infty$, and, therefore, no real root can lie between 1 and $+\infty$.

| 0 | and $-1$ |
|---|---|
| $-$ | $-$ |
| $-$ | $+$ |
| $+$ | $-$ |
| $+$ | $+$ |

Two variations lost, and there are two negative roots, between — and 0. We have, then, 3 roots, 1 positive and 2 negative. The signs being the same for $-1$ and $-\infty$ we know that plus 1 and minus 1 are the smallest limits in whole numbers.

Example 2. $x^3-4x^2-6x+8=0$.

$$\text{Here } V = x^3-4x^2-6x+8$$
$$V_1 = 3x^2-8x-6$$

Multiplying $V$ by the positive factor 3, and proceeding as above indicated.

$$\begin{array}{l|l}
3x^3-12x^2-18x+24 & 3x^2-8x-6 \\ \hline
3x^3-\ \ 8x^2-6x & x,-1 \\ \hline
-4x^2-12x+24; & \text{suppress } +4 \text{ and multiply by } 3: \\
-3x^2-\ \ 9x+18 & \\
-3x^2+\ \ 8x+\ \ 6 & \\ \hline
-17x+12 & \therefore\ V_2 = 17x-12
\end{array}$$

$$\begin{array}{l|l}
3x^2-8x-6 & 17x-12 \\ \hline
17 & 3x,-100 \\ \hline
51x^2-136x-102 & \\
51x^2-\ \ 36x & \\ \hline
-100x-102 & \\
\quad 17 & \\ \hline
-1700x-1734 & \\
-1700x-1200 & \\ \hline
\qquad -534 &
\end{array}$$

Then we have:

$$\begin{aligned}
V &= x^3-4x^2-6x+8 \\
V_1 &= 3x^2-8x-6 \\
V_2 &= 17x-12 \\
V_3 &= +534.
\end{aligned}$$

For $x=+\infty$, $+\ +\ +\ +$, no variations.
For $x=-\infty$, $-\ +\ -\ +$, 3 variations.
$3-0=3$ variations lost, and 3 real roots.

| | $V$ | $V_1$ | $V_2$ | $V_3$ | Var. |
|---|---|---|---|---|---|
| For $x=0$, | + | — | — | + | 2 |
| $x=1$, | — | — | + | + | 1 |
| $x=2$, | — | — | + | + | 1 |
| $x=3$, | — | — | + | + | 1 |
| $x=4$, | — | + | + | + | 1 |
| $x=5$, | + | + | + | + | 0 |
| $x=0$, | + | — | — | + | 2 |
| $x=-1$, | + | + | — | + | 2 |
| $x=-2$, | — | + | — | + | 3 |

Between —2, which gave the same signs as $-\infty$, and —1, there was 1 variation lost; hence the root is $-2+$ a fraction, or a root whose initial figure is —1. Between 0 and +1 a variation was lost, and there is a root whose initial figure is 0. At 4 there was 1 variation and at 5 none; hence a root 4 + a fraction.

Ex, 3. $2x^4 - 11x^2 + 8x - 16 = 0$. Here

$$V = 2x^4 - 11x^2 + 8x - 16$$
$$V_1 = 4x^3 - 11x + 4$$
$$V_2 = 11x^2 - 12x + 32.$$

If this were placed equal to zero the two roots would be found to be imaginary. If the trinomial were a true square we should have $4(11x^2+32) = (-12x)^2$ which it is not. $V_2$ will not change sign for any real value of $x$ and we will proceed no further in getting Sturm's functions.

$+\infty$ gives + + +; no variation

$-\infty$ gives + — +; 2 variations $\therefore$ $2-0=2$ and there are 2 real roots, and of course 2 imaginary roots.

| | | | |
|---|---|---|---|
| $x=0$ gives | — + + | $x=0$ gives | — + + |
| $x=1$ gives | — — + | $x=-1$ gives | — + + |
| $x=2$ gives | — + + | $x=-2$ gives | — — + |
| $x=3$ gives | + + + | $x=-3$ gives | + — + |

and the initial figures of the real roots are 2 and — 2.

Ex. 4. $x^3 + 11x^2 - 102x + 181 = 0$; *in which two of the roots are nearly equal.*

The functions are
$$\begin{aligned} V &= x^3 + 11x^2 - 102x + 181 \\ V_1 &= 3x^2 + 122x - 102 \\ V_2 &= 122x - 393 \\ V_3 &= + \text{ number.} \end{aligned}$$

$-\infty$ gives 3 variations and $+\infty$ none; so there are 3 real roots. $x = 0$ gives $+ - - +$ and so do $x = 1$, 2 and 3, but $x = 4$ gives no variation, therefore there are two positive roots lying between 3 and 4. Their initial figures are 3.

Let the equation be transformed into one whose roots are less by 3 (Art. 45). The functions of this equation will be:

$$\begin{aligned} V &= y^3 + 20y^2 - 9y + 1 \\ V_1 &= 3y^2 + 40y - 9 \\ V_2 &= 122y - 27 \\ V_3 &= + \text{ number.} \end{aligned}$$

Now in these substitute $y = 0$, $y = 0.1$, $y = 0.2$, $y = 0.3$, etc., and we find:

$$\begin{aligned} y &= 0 \text{ gives } + - - +, \quad \text{2 variations;} \\ y &= .1 \text{ gives } + - - +, \quad \text{2 variations;} \\ y &= .2 \text{ gives } + - - +, \quad \text{2 variations;} \\ y &= .3 \text{ gives } + + + +, \quad \text{no variation;} \end{aligned}$$

two positive roots between .2 and .3, and of the proposed equation between 3.2 and 3.3.

Transform the equation into another whose roots shall be less than the roots of the last by 0.2, and we have:

$$\begin{aligned} V &= s^3 + (20.6)s^2 - (\cdot 88)s + .008 \\ V_1 &= 3s^2 + (41.2)s - .88 \\ V_2 &= 122s - 2.6, \text{ or } 61s - 1.3 \\ V_3 &= + \text{ number.} \end{aligned}$$

$$\begin{aligned} \text{Substitute } s &= 0, \text{ the signs will be: } + - - +, \quad \text{2 var.} \\ s &= .01, \text{ the signs will be: } + - - +, \quad \text{2 var.} \\ s &= .02, \text{ the signs will be: } - - - +, \quad \text{1 var.} \\ s &= .03, \text{ the signs will be: } + + + +, \quad \text{no var.} \end{aligned}$$

One positive root between .01 and .02, also one between .02 and .03, and for $x$ we have $x = 3.21$ and $x = 3.22$. Their sum $= 6.43$. $\therefore$ $-11 - 6.43 = -17.43$, $=$ the third root, which is negative.

# CHAPTER VI.

## EXACT AND DIRECT SOLUTION OF EQUATIONS.

**Art. 84.** We know that equations of the first and second degrees admit of direct and exact solutions; the first presenting a single root and the latter two roots.

It will be now shown that such equations of the third degree as have *two imaginary roots* can be solved direcıly and exactly; and that equations of the 4th degree *of which two, and only two, of the roots are imaginary* can be directly and exactly solved. Above these equations there are no means of exact and direct solution, at least none as yet have been discovered, and it is believed that none can be discovered. Propositions for the exact solution of equations of degrees higher than the 4th have occasionally been presented, with plausibility, but the practical results have been such as not to invalidate the accuracy of the statement above.

**Art. 85.** Let $x^3 + Px^2 + Qx + R = 0$; this is a general representative of equations of the 3d degree, and any equation of the 3d degree will be a particular case of this general form.

But as the difficulties of solving equations of a degree higher than the second are sufficiently great at best it will be well to diminish them by removing as many terms as possible.

Now as we can exchange any complete equation for another wanting the second term, let us write

$$x^3 + px + q = 0 \ldots\ldots\ldots(1)$$

and form for it the functions of Sturm, we shall have

$$\begin{array}{l|l}
V = x^3 + px + q & x^3+px+q \mid \underline{3x^2+p} \\
V_1 = 3x^2 + p & \underline{3 \qquad\qquad} \mid \underline{x \qquad} \\
 & 3x^3+3px+3q \\
V_2 = -2px - 3q & \underline{3x^3+px \qquad} \\
V_3 = -4p^3 - 27q^2 & \qquad 2px+3q \therefore V_2 = -2px-3q
\end{array}$$

Again:

$$
\begin{array}{l|l}
3x^2+p & -2px-3q \\
2p & \quad -3x,\ 9q \\
\hline
6px^2+2p^2 & \\
6px^2+9qx & \\
\hline
\quad -9qx+2p^2 & \\
\quad\quad 2p & \\
\hline
\quad -18pqx+4p^3 & \\
\quad -18pqx-27q^2 & \\
\hline
\quad 4p^3+27q & \\
\quad \therefore V_3=-4p^3-27q^2 &
\end{array}
$$

The roots of this equation, it being of the 3d degree, must all be real, or else one must be real and two imaginary. Since $p$ and $q$ are not numbers, but the general representatives of any real coefficients of the first and zero powers, let us see what relations they must have in order that *all the roots shall be real.* There must result from the substitution of $-\infty$ 3 variations, and from the substitution of $+\infty$ no variations, in order that all the roots may be real. When $+\infty$ is substituted in $V$ the result is $+$; in $V_1$ it is $+$, but in $V_2$ it will not be $+$ *unless $p$ is negative.* Then $p$ must be negative; but $V_3$ will not be $+$ unless $p$ is negative and moreover has such a value that $4p^3$ is greater than $27q^2$. With this condition $V_3$ will be plus and remain so. The first term of $V_2$ will also have a positive coefficient; so that the substitution of $-$ will give the signs $-+-+$; 3 variations, and the substitution of $+\infty$ gives no variations.

We see, then, when $\frac{p^3}{27}>\frac{q}{4}$ or $\left(\frac{p}{2}\right)^3>\left(\frac{q}{2}\right)^2$ that all the roots will be real.

## CARDAN'S SOLUTION OF EQUATIONS OF THE THIRD DEGREE.

**Art. 86.** If the equation is complete, let it be transformed to another wanting the second term. It will take the form

$$x^3+px+q=0, \ldots\ldots\ldots\ldots(1)$$

Let the unknown quantity be placed equal to the sum of two other unknown quantities, $y$ and $z$; then $x = y+z$; $x^3 = y^3+2yz(y+z)+z^3$ $\therefore$ $x^3-3y(y+z)z-(y^3+z^3) = 0$, and replacing $y+z$ by $x$ in the second term, we have:

$$x^3-3yzx-(y^3+z^3) = 0, \ldots\ldots\ldots\ldots (2)$$

This has the same form as eq. (1), and by comparison:

$p = -3yz, \ldots (3)$, and $q = -(y^3+z^3)$, or $y^3+z^3 = -q, \ldots (4)$

Since $z = \frac{-p}{3y}$, $z^3 = \frac{-p^3}{27y^3}$, and this in the value of $-q$ gives $y^3-\frac{p^3}{27y^3} = -q$; $\therefore$ $y^6+qy^3 = \frac{p^3}{27}$.

Since this is a trinomial equation, we have:

$$y = \sqrt[3]{-\frac{q}{2}\pm\sqrt{\left(\frac{q^2}{4}+\frac{p^3}{27}\right)}}$$

and, since the equation $y+z = x$ is symmetrical with respect to $y$ and $z$, $z$ will have the same values. But not to repeat the same value for both, we will take the first for $y$ and the second for $z$, and adding them together, we get for the value of $x$:

$$x = \sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q^2}{4}+\frac{p^3}{27}\right)}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q^2}{4}+\frac{p^3}{27}\right)}}$$

which is the celebrated Formula of Cardan.

Under each grand radical sign there is the same indicated square root, which will be real when $p$ is positive, and also when $p$ is negative, provided that $\frac{p^3}{27}$ is less than $\frac{q^2}{4}$. $\frac{q^2}{4}$ is of course always positive. But the inequation $\frac{p^3}{27}<\frac{q^2}{4}$, by Art. 78, shows that all the roots cannot be real, therefore *one will be real and two imaginary*. Cardan's formula will then solve the cubic equation in such a case.

If the last inequation should prove to be an equation in

any case, then the indicated square root would vanish and $x = 2\sqrt[3]{-\frac{q}{2}}$; which is real, and Cardan's formula would apply in this case also. But if we seek the greatest common divisor between the first member of the equation and its first derived polynomial, the remainder is $4p+27q^2$, and if this $= 0$, there is a C.D. of the first degree with respect to $x$, and therefore *two of the roots are equal.*

If *all three of the roots were equal*, the equation would reduce to a binomial equation of the form $(x-a)^3 = 0$, and $x = a$, and all the roots would be $= a$.

But when $p$ is negative and $\frac{p^3}{27} > \frac{q^2}{4}$, Cardan's Formula fails, and this is the *condition, that all the roots are real*, but not equal. All are unequal.

**Art. 87.** Since every quantity has 3 cube roots, and since $x =$ sum of two cube roots, it might at first sight appear that the cubic equation had *nine roots*, because each one of the first set might be taken in conjunction with each of the 3 in the second set, making 9 in all. Now we know that every equation of the third degree has three roots and no more, and this appearance of nine roots must be deceptive.

To explain this, let us remember that the three cube roots of any quantity, as $a^3$ are of the form $a$, $\frac{a(1+\sqrt{-3})}{2}$ and $\frac{a(1-\sqrt{-3})}{2}$, and so of the values of $y$ and $z$. But it will be remembered (eq. 3, Art. 79), that $yz = -\frac{p}{3}$, a real quantity, since $p$ and $q$, the coefficients of the proposed equation, are supposed to be real. If, then, we take one of the cube roots which are equal to $y$ and one of the cube roots which are equal to $z$ to make a value of $x$, that is, to make up a root of the equation, we must so take them that their product will be real. We could take the two rational

and real cube roots, but when we take any one of the imaginary expressions we must take another, but differing in sign before the radical part of it, so that the product would be the difference of two squares and real. This restriction would allow us to form two values only of $x$ out of the imaginary parts, which with the one made up of the real parts would give the number of values of $x$, three, and only three.

In the solution of numerical examples, it will in general suffice to substitute in Cardan's Formula for $q$ and $p$ their appropriate values; but sometimes greater simplicity may be obtained by treating the example in the same manner that $x^3+px+q=0$ was treated in the deduction of Cardan's Formula.

EXAMPLES.

1. Solve the equation $x^3-9x^2+28x-30=0$.

Ans. $3$, $3+\sqrt{-1}$, $3-\sqrt{-1}$.

The transformed equation is $u^3+u=0$, in which $p=1$ and $q=0$ $\therefore$ Cardan's Formula becomes:

$u=$

$$\sqrt[3]{\sqrt{\left(\frac{1}{27}\right)}}+\sqrt[3]{-\sqrt{\left(\frac{1}{27}\right)}}; \text{ but } \sqrt[3]{-\sqrt{\left(\frac{1}{27}\right)}}=\sqrt[3]{(-1).\sqrt{\left(\frac{1}{27}\right)}}=$$

$-1.\sqrt[3]{\sqrt{\left(\frac{1}{27}\right)}}$ $\therefore$ $u=0$, one root, and as $x=u+3$, $x=3$. This "divided out" of the equation leaves an equation of the second degree with roots as above.

2. Solve the equation $x^3-7x^2+14x-20=0$.

Ans. $5$, $1+\sqrt{-3}$, $1-\sqrt{-3}$.

The transformed equation is $u^3-\frac{7}{3}u-\frac{344}{27}=0$.

And this cleared of fractions gives: $y^3-21y-344=0$. One root of this is 8 and $u=\frac{y}{3}=\frac{8}{3}$; but $x=u+\frac{7}{3}=\frac{8+7}{3}$ $=5$.

Ex. 3. Solve $x^3 - 6x^2 + 10x - 8 = 0$.

Here $u^3 - 2u - 4 = 0$, and the two radicals in Cardan's formula give $1.577+\ldots$ .and $0.422+$; their sum is $2 = u$.

Ex. 4. $x^3 + 2x + 12 = 0$.

Ex. 5. $x^3 - 48x = 128$.

Ex. 6. $x^6 - 3x^3 - 2x^2 - 8 = 0$. Let $x^2 = t$.

REMARK—When $\frac{q^2}{4} < \frac{p^3}{27}$ and the latter is negative, the values of $x$ are *apparently imaginary* though known in fact to be all real. This is called the *irreducible case* because Cardan's formula fails, and no means have yet been discovered to surmount the difficulty by Algebra alone.

### SOLUTION OF EQUATIONS OF THE FOURTH DEGREE.

**Art. 88.** The equations of this degree admit of direct and exact solution by the methods now known only when they have *two, and only two, imaginary roots*. If their roots are all real or are all imaginary we do not know how to solve them exactly, but in the case of numerical equations can resort to some method of approximation.

Descartes and Waring have each demonstrated an excellent method of solving equations of the fourth degree having only two imaginary roots. We will here reproduce

### DR. WARING'S METHOD OF EXACT SOLUTION OF EQUATIONS OF THE 4TH DEGREE.

**Art. 89.** Let the proposed equation be

$$x^4 + 2px^3 = qx^2 + rx + s \ldots\ldots (1)$$

Now $(x^2 + px + n)^2 = x^4 + 2px^3 + (p^2 + 2n)\,x^2 + 2pnx + n^2 \ldots\ldots (2)$

If therefore we should add to both members of eq (1) the quantity $(p^2 + 2n)\,x^2 + 2pnx + n^2$ the first member would be a perfect square.

The second member becomes

$(p^2 + 2n + q)\,x^2 + (2xn + r)\,x + (n^2 + s)$; which, being in reality a trinomial arranged according to the descending powers of $x$, will be a perfect square if 4 times the product of the extreme terms = the square of the middle term;

that is if $4(p^2+2n+q)(n^2+s)=(2pn+r)^2$; leaving off for the moment the powers of $x$. Performing the operations indicated, transferring all the terms to the first member and arranging according to the undetermined quantity $n$ we get

$$8n^3+4qn^2+(8s-4pr)n+4qs+4p^2s-r^2=0\ldots(3);$$

an equation of the 3d degree with respect to $n$. If we can solve this conditional cubic equation and get a value for $n$, this value of $n$ will be what is required to make the trinomial $(p^2+2n+q)x^2+(2pn+r)x+(n^2+s)$ a perfect square. The quantity involving $n$ which was added to both members made the first member a perfect square, and with the value of $n$ (which we still call $n$) supposed to have been found from the cubic (3) both members will be exact squares:

$$(x^2+px+n)^2=(p^2+2n+q)x^2+(2pn+r)x+(n^2+s);$$

taking their square roots we have

$x^2+px+n=\pm\left[\sqrt{p^2+2n+q}.\ x+\sqrt{n^2+s}\right]$ when the middle term of the trinomial which makes the second member is positive; and when that middle term is negative we have $x^2+px+n=\pm\left[\sqrt{p^2+2n+q}.x-\sqrt{n^2+s}\right]$.

In either case we have two equations of the 2d degree and will get 4 values of $x$, which are the 4 roots of the original equation.

**Art. 90.** This method can be applied only to those equations of the 4th degree which have two imaginary and two real roots. For let us suppose the roots to be represented by $a$, $b$, $c$ and $d$; the product of two of them, say $ab$, will equal the absolute term of one of the quadratics, giving $ab=n-\sqrt{n^2+s}$; $\therefore\ n-ab=\sqrt{n^2+s}$ squaring which we get

$$n^2-2abn+a^2b^2=n^2+r\ \therefore\ -2abn+a^2b^2=s,$$

but as $s$, the absolute term of the *proposed* equation, is equal

to $-abcd$, we have $-2abn + a^2b^2 = -abcd$ $\therefore$ $ab - 2n = -cd$ $\therefore n = \dfrac{ab + cd}{2}$. Similarly the other two values of $n$ are $n = \dfrac{ac + bd}{2}$ and $n = \dfrac{ad + bc}{2}$

If now *all the roots of the bi-quadratic, or equation of the 4th degree, a, b, c and d should be real,* the values of $n$, or the roots of the equation of the 3d degree would all three be real and we could not solve it.

Again if $a$, $b$, $c$, $d$ should all be imaginary, their products, two and two, being real, the 3 values of $n$ would all be real and we could not solve. But if two of the roots $a$, $b$, $c$ and $d$ are real and two imaginary there will be one root of the cubic real and two imaginary and then Cardan's Formula would apply. Thus suppose that $a$ and $c$ were real, $b$ and $d$ imaginary; the value $n = \dfrac{ac + bd}{2}$ would be real because we should have the product of two real + the product two imaginary quantities, and the sum would be real. The other roots would evidently be imaginary.

### EXAMPLE.

$x^4 - 6x^3 + 5x^2 + 2x - 10 = 0$; by comparing this with the formula equation we find

$2p = -6$ or $p = -3$

$q = -5$

$r = -2$

$s = \ \ 10$ and equation (3) then becomes

$8n^3 - 20n^2 + 56n + 156 = 0$, which divided by 4 gives $2n^3 - 5n^2 + 14n + 39 = 0$. On solving this we find one root $n = -\dfrac{3}{2}$. Hence

$\left(x^2 - 3x - \dfrac{3}{2}\right) = x^2 + 7x + \dfrac{49}{4}$ $\therefore x^2 - 3x - \dfrac{3}{2} = \pm\left(x + \dfrac{7}{2}\right)$

$\therefore x^2 - 4x \quad 5$ and also, the other quadratic, $x^2 - 2x = -2$: From these we get $x = -1, 5, 1 + \sqrt{-1}$ and $1 - \sqrt{-1}$.

The solution of the intermediate cubic results thus:

$n^3 - \frac{5}{2}n^2 + 7n + \frac{39}{2} = 0$; make $n = \frac{y}{2} \therefore y^3 - 5y^2 + 28y + 156 = 0$; make $y = t + \frac{5}{3} \therefore t^3 + \frac{59}{3}t + \frac{5222}{27} = 0$. Make $t = \frac{s}{3} \therefore s^3 + 177s + 5222 = 0$. In this last equation for the purpose of applying Cardan's formula we note $p = 177$ and $q = 5222$, hence $\frac{p^3}{27} = 205379$ and $\frac{q^2}{4} = 6817321$ and the algebraic sum of the two cube roots in the formula $= -14 = s$. $\therefore t = \frac{s}{3} = -\frac{14}{3}$ and $y = t + \frac{5}{3} = -\frac{14}{3} + \frac{5}{3} = -3 \therefore n = \frac{y}{2} = -\frac{3}{2}$.

EXAMPLE 2.

Find the roots of $x^4 - x^3 - 17x^2 - 3x - 60 = 0$

$n = -\frac{17}{2}$. Ans. $-4,\ 5,\ \sqrt{-3},\ -\sqrt{-3}$.

EXAMPLE 3.

Find the roots of $x^4 + 7x^3 - 33x^2 + 107x - 154 = 0$.

$n = -\frac{15}{2}$. Ans. $2,\ -11,\ 1+\sqrt{-6},\ 1-\sqrt{-6}$.

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## CHAPTER VII.

### Occasional Solution of Higher Equations.

**Art. 91.** Beyond equations of the fourth degree there are no direct methods for exact solution; and as has been seen, the existing methods do not apply to all equations of the third and fourth degrees. But when equations of any degree

HAVE EQUAL ROOTS

those equal roots may be discovered by Art. 53, and divided out, thus reducing the degree of the equation. If the reduced degree is the first or second, the remaining root or roots may be directly found; if the reduced degree is the third and the equation has two imaginary roots, Cardan's Formula will apply and the equation may be directly solved; also when the resulting equation is of the fourth degree and has two, and only two, imaginary roots, the equation may be exactly solved.

**Art. 92.** Also when an equation

OF THE THIRD DEGREE HAS COEFFICIENTS WITH CERTAIN SPECIAL RELATIONS,

the roots may be found exactly.

Suppose the equation to be of the form

$$x^3 + dx^2 + kx = q \;\ldots.(1)$$

If we had a cubic equation of the following form

$$x^3 + 3px^2 + 3p^2x = q \;\ldots.(2)$$

it is evident upon inspection that if we add $p^3$ to the first member it would become a perfect cube. Adding $p^3$ then to both members we have

$$x^3 + 3px^2 + 3p^2x + p^3 = q + p^3 \ldots.(3) \text{ or}$$

$$(x + p)^3 = q + p^3 \text{ whence } x = -p + \sqrt[3]{q+p^3} \ldots.(4)$$

Comparing equations (1) and (2) we see that $d = 3p$ and $k = 3p^2$, whence $p = \frac{d}{3}$ and $p^2 = \frac{k}{3}$. Squaring both members of $p = \frac{d}{3}$ we have $p^2 = \frac{d^2}{9}$ $\therefore$ by addition, $2p^2 = \frac{d^2}{9} + \frac{k}{3} = \frac{d^2+3k}{9}$ $\therefore$ $p = \pm\sqrt{\frac{3k+d^2}{18}}$ ......(5)

From this last formula the value of $p$ may be found, which is necessary to transform equation (1) into equation

(2) and which in (4) will give a root of the proposed equation.

But this is on condition that $p$ have the same value in $d=3p$ and $k=3p^2$, that is, that $d^2=3k$. When the coefficients of $x^2$ and $x$ are such that the square of the coefficient of $x^2$ is equal to three times that of $x$, this method will apply.

EXAMPLE 1.

$$x^3+15x^2+75x=-125.$$

Here $d=15$ and $d^2=225$, and $3k=3\times75=225$; and $p=\sqrt{\frac{225+225}{18}}=5$, and $x=-5+\sqrt[3]{-125+125}=-5$, and this root divided out gives the quadratic $x^2+10x+25=0$, of which the roots are $-5$ and $-5$. All three of the roots are real and equal in this equation.

EXAMPLE 2.

$$x^3+15x^2+75x=218.$$

Here $p=\frac{d}{3}=5$, and $x=-p+\sqrt[3]{q+p^3}=-5+\sqrt[3]{218+125}=-5+\sqrt[3]{243}=-5+7=2.$

The other roots are:

$$x=\frac{-17+\sqrt{-147}}{2} \text{ and } x=\frac{-17-\sqrt{-147}}{2}$$

EXAMPLE 3.

$$x^3+x^2+\frac{x}{3}=\frac{7}{3}.$$

Ans. $x=1$, $x=-1+\sqrt{-\frac{4}{3}}$, $x=-1-\sqrt{-\frac{4}{3}}$

It will be observed that when this relation holds between the coefficients, if the second term be made to disappear the third will disappear also. Further, that when applicable it is so without respect to the nature of the roots, as imaginary or not. If, then, the intermediate cubic in the

solution of an equation of the fourth degree should be of the class just described, it would enable us to solve that equation without respect to the nature of its roots, widening by so much the field of application of Waring's and Descartes' methods.

### WHOLE-NUMBER ROOTS.

**Art. 93.** If an equation has been placed in the reduced form, $x^m + Px^{m-1} + Qx^{m-2} + \ldots + Tx + U = 0$, (1) it cannot have any roots which are fractions.

By fractions is here meant irreducible fractions, or fractions whereof the two terms are prime with respect to each other. Such fractions containing "one or more of the equal parts of unity" are commensurable with unity. Whole nnmbers are, of course, commensurable with unity. See Art. 26.

**Art. 94.** Consequently, when we find no whole number among the roots of an equation of the reduced form, since we already know that there are no fractions among them, we know that none of the roots are commensurable with unity. $3+\sqrt{2}$, $\sqrt[3]{7}$, are specimens of quantities not commensurable with unity. Imaginary quantities are never commensurable with unity.

**Art. 95.** The absolute term of the equation will contain as divisors all the roots, whether whole numbers or not; but it will usually contain many other divisors besides the roots. We could scarcely hope to find what the incommensurable divisors of any absolute term are, bnt we can more easily discover those which are whole numbers; and of these, taking those lying between a superior limit of the positive roots, $L$, and a numerically superior limit of the negative roots, $-L''$, we can discover by trial which among them are roots.

But the labor of these substitutions may be much shortened by the results of the following investigation.

**Art. 96.** Let $a$, a whole number, be a root. Then

$$a^m+Pa^{m-1}+Qa^{m-2}+\ldots.+Ta+U=0;$$

and transposing to the second member all the terms except $U$, and dividing by $a$, we have:

$$\frac{U}{a}=-a^{m-1}-Pa^{m-2}-\ \ldots.\ -Ra^2-Sa-T.\ \ldots..(1)$$

Since the second member contains none but whole numbers, it is entire, and therefore $\frac{U}{a}$ is entire, which is merely confirmatory of what we already knew. Now transposing $T$ to the first member, and dividing by $a$, we obtain:

$$\frac{U}{a}+T=-a^{m-2}-Pa^{m-3}-\ldots.-Ra-S,$$

and as the second member is entire, we see that *the quotient of the absolute term divided by the whole number root, plus the coefficient of $x$ is also exactly divisible by that root.*

For $\frac{U}{a}+T$ substitute $T'$, transpose $S$ and divide by $a$ as before, and the result is:

$$\frac{T'+S}{a}=-a^{m-3}-Pa^{m-4}-\ \ldots.\ -Qa-R,$$

and as this second member is entire, we see that *the former quotient plus the coefficient of $x^2$ is exactly divisible by the root.*

Now making $\frac{T'+S}{a}=S'$, transposing $-R$ and dividing by $a$ as before, there results:

$$\frac{S'+R}{a}=-a^{m-4}-Pa^{m-5}-\ldots\ldots.-Q,$$

a whole number. *Therefore the last preceding coefficient plus the coefficient of $x^3$ is exactly divisible by the root.*

Proceeding in the same manner, when we shall have transposed all the terms save two, we shall have an equation like $\frac{Q'}{a}=-a-P=$ a whole number. Transposing $-P$ and dividing as before: $\dfrac{\frac{Q'}{a}+P}{a}=\frac{P'}{a}=-1.$ This shows

that the last of the quotients (which is formed when the coefficient of the second term is transposed) is $-1$. Every divisor of the absolute term which will stand all of these successive tests is a root, and as it is supposed that we will try only those divisors which are whole numbers, we will discover all the whole-number roots.

Having found them we divide them out, as in the case of equal roots, and solve, if possible, the resulting equation.

We may form a table at the heads of the vertical columns of which are placed all the entire divisors which lie between the upper and lower limits, and then make a simultaneous trial of them all; rejecting all which in any of the successive divisions give quotients not entire; that is, any which fail to stand all the tests.

Having formed the table by writing the whole-number divisors between the limits in a horizontal row proceed by the

RULE

*Divide the absolute term by each divisor setting the quotient immediately beneath the divisor. Form new dividends by adding the coefficient of $x$ to the quotients. Divide these by the numbers on trial, setting the quotients immediately beneath the dividends. Form new dividends by increasing the last quotients by adding to them the coefficient of $x^2$. So proceed, always forming new dividends by the addition to the last quotients of the next succeeding coefficient towards the first, and rejecting any divisor which at any stage gives a fractional quotient. All those which finally give a quotient which is minus unity are roots.*

EXAMPLE 1.

Find the entire roots of the equation:

$$9x^6+30x^5+22x^4+10x^3+17x^2-20x+4=0.$$

$$x^6+\frac{10x^5}{3}+\frac{22}{9}x^4+\frac{10}{9}x^3+\frac{17}{9}x^2-\frac{20}{9}x+\frac{4}{9}=0;\text{ make } x=\frac{y}{3}\therefore$$

$$y^6+10y^5+22y^4+30y^3+153y^2-540y+324=0.$$

Here $L=4$ and $-L''=-31$

And the divisors (which are whole numbers) of the absolute

term 324, are 4, 3, 2, 1, —1, —2, —3, —4, —6, —9, —12, —18 and —27.

| 3 | 2 | 1 | —1 | —2 | —3 | —4 | —6 | —9 | —12 | —18 | —27 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 108 | 162 | 324 | —324 | —168 | —108 | —81 | —54 | —36 | —27 | —18 | —12 |
| —432 | —378 | —216 | —864 | —708 | —648 | —621 | —594 | —576 | —567 | —558 | —552 |
| —144 | —189 | —216 | +864 | +354 | +216 | $+155\frac{1}{4}$ | +99 | +64 | $+47\frac{3}{12}$ | +31 | $+20\frac{12}{27}$ |
| 9 | —36 | —63 | 1017 | 507 | 369 | | 252 | 217 | | 184 | |
| +3 | —18 | —63 | —1017 | $-253\frac{1}{2}$ | —123 | | —42 | $-24\frac{1}{9}$ | | $-10\frac{2}{9}$ | |
| 33 | 12 | —33 | —987 | | —93 | | —12 | | | | |
| +11 | +6 | —33 | +987 | | +31 | | +2 | | | | |
| 33 | 28 | —11 | 1009 | | 53 | | 24 | | | | |
| +11 | +14 | —11 | —1009 | | $-17\frac{2}{3}$ | | —4 | | | | |
| 21 | 24 | —1 | —999 | | | | +6 | | | | |
| +7 | +12 | —1 | +999 | | | | —1 | | | | |

The only two divisors giving a final result of $-1$ are $+1$ and $-6$ and placing these in $x = \frac{y}{3}$ we have $x = \frac{1}{3}$ and $x = -2$ which will satisfy the equation proposed.

2. $x^5 - 2x^4 - 15x^3 + 8x^2 + 68x + 48 = 0$.
Ans. $-2, -2, -1, 3$ and $4$.

3. $x^4 - 5x^3 + 25x - 21 = 0$.
Ans. $3, 1, \frac{1+\sqrt{29}}{2}$ and $\frac{1-\sqrt{29}}{2}$

SOLUTION OF RECURRING EQUATIONS.

**Art. 97.** These equations have their coefficients recur when counted from the first and last. Their roots are of the form $a, \frac{1}{a}, b, \frac{1}{b}$, etc.

The student should now carefully review Articles 41 to 43, inclusive.

*Binomial equations* of the form $x^3+1=0$, $x^4+1=0$, $x^4-1=0$, etc., are recurring equations.

1. Take $x^3-1=0$; we know that $+1$ is a root of this equation; dividing it out, we get $\frac{x^3-1}{x-1} = x^2+x+1$, and the roots of $x^2+x+1=0$ are $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$.

2. Take $x^3+1=0$; we know that $-1$ is a root of this equation, and dividing it out we get $\frac{x^3+1}{x+1}=x^2-x+1$; the roots of $x^2-x+1=0$ are $\frac{-1+\sqrt{-3}}{2}$ and $\frac{-1-\sqrt{-3}}{2}$.

3. Take $x^4-1=0$; we know that this is composed of two factors of the second degree, to wit, $x^2-1$ and $x^2+1$; hence placing these equal to zero and solving, we get the four roots, $+1$, $-1$, $+\sqrt{-1}$ and $-\sqrt{-1}$.

4. Take $x^6-1=0$; this is $(x^3+1)(x^3-1)=0$, giving cases 1 and 2.

5. $x^5+1=0$ is $(x+1)(x^4-x^3+x^2-x+1)=0$, giving $x+1=0$ and $x^4-x^3+x^2-x+1=0$. This last is a recurring equation of the fourth degree.

But before examining $x^4+1=0$, $x^6+1=0$, etc., we will demonstrate the following principle:

**Art. 98.** *Every recurring equation of the fourth and higher even degrees may be solved by using one of a degree half as high.*

Suppose we had a recurring equation of the fourth degree:

$$x^4+Px^3+Qx^2+Px+1=0,$$

and that the roots were $a$, $\frac{1}{a}$, $b$ and $\frac{1}{b}$; then the factors of the first degree would be $x-a$, $x-\frac{1}{a}$, $x-b$ and $x-\frac{1}{b}$, and the quadratic factors would be $x^2-\left(a+\frac{1}{a}\right)x+1$ and $x^2-\left(b+\frac{1}{b}\right)x+1$. Put $a+\frac{1}{a}=k$ and $b+\frac{1}{b}=l$; then we have $x^2-kx+1$ and $x^2-lx+1$. These multiplied together give the equation of the fourth degree in which $x$ enters as a factor four times, while $k$ enters only twice; therefore whatever equation we may obtain for the value of $k$, from or by

means of the original equation, will be only of the second degree. If we multiply the quadratic factors involving $k$ and $l$, and place the product equal to the original equation, we will form an identical equation, and equating the corresponding coefficients, we could determine the values of $k$ and $l$; and it would be found that none of the subordinate equations would be above the second degree.

Again, suppose the recurring equation to be of the sixth degree and put $k=a+\frac{1}{a}$, $l=b+\frac{1}{b}$ and $h=c+\frac{1}{c}$. The quadratic factors $x^2-kx+1$, $x^2-lx+1$ and $x^2-hx+1$ multiplied together give $x^6-(k+l+h)x^5+(kl+kh+lh+3)x^4-(klh+2k+2l+2h)x^3+(kl+kh+lh+3)x^2-(k+l+h)x+1$.

Put this equal to the first member of original equation, supposed to be $x^6+Px^5+Qx^4+Rx^3+Qx^2+Px+1$, and equating the coefficients, we get:

$$P=-(k+l+h);$$
$$Q=kl+kh+lh+3;$$
$$R=-(klh+2k+2l+2h);$$

in which, since $P$, $Q$, and $R$ are known numbers, we have to determine the three unknown quantities, $k$, $l$ and $h$ from the three equations, one being of the first, one of the second and one of the third degree. The resulting equation would therefore be of the third degree, one-half the degree of the original equation.

A similar investigation would evidently show a similar result for any equation of a higher and even degree.

Returning to Example 5—we there saw that one of the roots was $-1$, and when this was divided out there resulted the recurring equation $x^4-x^3+x^2-x+1=0=x^4-(k+l)x^3+(kl+2)x^2-(k+l)x+1$. From this $1=k+l$ and also $1=kl+2$ $\therefore$ $k^2-k=1$ and $k=\frac{1\pm\sqrt{5}}{2}$. But as $a+\frac{1}{a}=k$,

$$a=\frac{k\pm\sqrt{k^2-4}}{2}=\frac{\frac{1\pm\sqrt{5}}{2}\pm\sqrt{\frac{3\pm\sqrt{5}}{2}-4}}{2}=$$

$$\frac{1\pm\sqrt{5}\pm\sqrt{-10+2\sqrt{5}}}{4}$$

and either of these four roots when substituted in $x^5+1=0$, will satisfy it.

For instance take the first: $\dfrac{1+\sqrt{5}+\sqrt{-10+2\sqrt{5}}}{4}$ and raise it to the fifth power. $4^5=1024$. In the numerator place $1+\sqrt{5}=c$ and $\sqrt{-10+2\sqrt{5}}=d$. Then $(c+d)^5=c^5+5c^4d+10c^3d^2+10c^2d^3+5cd^4+d^5$ and

$$\begin{aligned}
c^5 &= 176+80\sqrt{5} \\
5c^4d &= +280\sqrt{-10+2\sqrt{5}}+120\sqrt{5}\sqrt{-10+2\sqrt{5}} \\
10c^3d^2 &= -800-480\sqrt{5} \\
10c^2d^3 &= -400\sqrt{-10+2\sqrt{5}}-80\sqrt{5}\sqrt{-10+2\sqrt{5}} \\
5cd^4 &= -400-400\sqrt{5} \\
d^5 &= +120\sqrt{-10+2\sqrt{5}}-40\sqrt{5}\sqrt{-10+2\sqrt{5}} \\
\hline
& -1024
\end{aligned}$$

—1024, and this divided by the fifth power of the denominator gives —1, which satisfies the equation. In a similar manner other recurring equations, like $x^6+1=0$ and $x^m+1=0$, may be solved whenever we can solve an equa-

tion of half the degree, and this is true of all equations which are recurring, whether binomial or not.

### EXPONENTIAL EQUATIONS.

**Art. 99.** Exponential equations, or such as have the *unknown quantity as an exponent*, sometimes, but rarely, admit of exact solution. It is assumed that the student is familiar with the solution of exponential equations by continued fractions and logarithms.

These equations do not fall within the class of algebraic equations, but of transcendental equations.

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## CHAPTER VIII.

### APPROXIMATE SOLUTIONS OF HIGHER NUMERICAL EQUATIONS.

**Art. 100.** When it is not practicable to solve an equation by any of the modes which have been discussed, we must rest content with an approximation to the roots. Fortunately this can be had closely enough, for practical purposes, by the methods which we now propose to examine.

We have seen that when an equation has some equal roots, or some that are whole numbers, we may discover them and divide them out, and reduce the degree of the equation; if it is a recurring equation, we shall have to solve one only half as high in degree; and in short, if by trial, chance or in any other way, we can discover one or more roots, we would immediately depress the degree. If after all it is of the fifth or higher degree, we can only approximate, and so likewise with those of the third and fourth degrees when they do not happen to have two imaginary roots and no more; unless there is a peculiar relation among the coefficients such as was examined in Art. 92.

### HORNER'S METHOD.

**Art. 101.** This method was first published in 1819. It is the invention of W. G. Horner, and is regarded by most

mathematicians as the most satisfactory mode of approximating to the real and incommensurable roots of an equation having numerical coefficients. The method is as follows:

1st. Having found by Sturm's Theorem, or otherwise, the whole-number part of a root, and still better, having found in addition one or more of the figures in the decimal part, to transform the original equation into another whose roots shall be less by the part already found.

2d. To obtain the next figure of the root by dividing the absolute term by the next preceding coefficient and taking the first figure of the quotient for the required figure.

3d. Then to transform this equation into another whose roots shall be less by the decimal figure last obtained; to divide the absolute term of this equation by the coefficient which immediately precedes it, and take the first figure of the quotient for the next figure of the root.

4th. Again transform the equation into another of which the roots shall be less by the decimal figure last obtained, divide the last coefficient by the one immediately preceding for the next figure of the root, and so continue till the desired number of places in the approximate root shall be found.

In this way we find the real positive roots, and if there are any which are negative, obtain them approximately by changing the alternate signs of the proposed equation, which will make the roots now being sought all positive, and proceed as before.

If preferred, when one or more roots have been found, they may be divided out and the degree of the equation reduced.

*Demonstration.*

**Art. 102.** When an equation has been transformed into another of which the roots are less by the whole-number part of the original root, and still more if they are less by the whole-number part and one or more figures of

the decimal part, the remainder of the root, that is, the value of the unknown quantity in the transformed equation, is a very small quantity indeed. Therefore its second and higher powers may be neglected in comparison with itself, and the first member of the transformed equation, which will be of the form

$$u^m + P'u^{m-1} + Q'u^{m-2} + \dots + T'u + U' = 0,$$

may be, without appreciable error, taken to be

$$T' + U' = 0; \;\therefore\; u = -\frac{U'}{T'};$$

and the first figure of the quotient (and perhaps more) will be the initial figure or figures of the true value of $u$, and will therefore be the next required figure in the value of the original unknown quantity.

Let it be required to find the approximate roots of the roots of the equation:

$$x^4 - 8x^3 + 14x^2 + 4x - 8 = 0.$$

Sturm's functions of this, when reduced to their simplest form, are:

| | $-\infty$ | $+\infty$ | 0 | $-1$ | $+1$ | $+2$ | $+3$ | $+5$ | $+6$ |
|---|---|---|---|---|---|---|---|---|---|
| $V = x^4 - 8x^3 + 14x^2 + 4x - 8$ | + | + | − | + | + | + | − | − | + |
| $V_1 = x^3 - 6x^2 + 7x + 1$ | − | + | + | − | + | − | − | + | + |
| $V_2 = 5x^2 - 17x + 6$ | + | + | + | + | − | − | ± | + | + |
| $V_3 = 76x - 103$ | − | + | − | − | − | + | + | + | + |
| $V_4 = +$ number | + | + | + | + | + | + | + | + | + |
| Variations | 4 | 0 | 3 | 4 | 2 | 2 | 1 | 1 | 0 |

Since the number of variations lost between $-1$ and $+6$ is the same as between $-\infty$ and $+\infty$, all the roots are real and comprised between $-1$ and 6.

Between $-1$ and 0 there is $4 - 3 = 1$ variation lost; there is one negative root, then, between 0 and $-1$, and since it is numerically less than $-1$, we substitute in $V$, $V_1$, $V_2$, etc., in succession, $-.1$, $-.2$, $-.3$, etc., until there

is a gain of variation, and the last preceding number will be the first figure of the root. This root in the example is found between —.7 ond —.8; therefore .7 is the first figure of the *negative root.*

0 gives 3 variations, and +1 gives 2; hence there is a *positive root* which is a decimal fraction. By the successive substitution of .1, .2, .3, etc., its first figure is found to be .7.

2 gives 2 variations and 3 gives 1; hence there is a root whose first figure is 2; and as 5 gives 1 variation and 6 gives none, there is a root whose first figure is 5. This last fact may be known at once, because 5 in $V$ gives a negative result and 6 in $V$ gives a positive result. There is therefore one real root between them, or else some other odd number of roots.

Let us now proceed after Horner's manner to find this root, whose whole-number part is 5. The coefficients of the original equation are:

| | | | | | |
|---|---|---|---|---|---|
| 1 | —8 | +14 | +4 | —8 | 5 |
| | +5 | —15 | —5 | —5 | |
| 1 | —3 | —1 | —1 | ,—13 | |
| | +5 | +10 | +45 | | |
| 1 | +2 | +9 | ,+44 | | |
| | +5 | +35 | | | |
| 1 | +7 | ,+44 | | | |
| | +5 | | | | |
| 1 | ,+12; | | | | |

and 13÷44 = .2....; .2 is the next figure in the root, and the coefficieuts of the transformed equation, that is, one whose roots are less than those of the original equation by 5, are: 1, +12, +44, +44, —13. It will be observed that they run in a diagonal line from left to right upwards in the calculation of the transformation.

Now let us get an equation whose roots are less by .2 than those of the equation whose coefficients are:

```
1+12    +44     +44      -13      | 2
   0.2 + 2.44 + 9.288 +10.6576
-----------------------------------
 +12.2 +46.44 +53.283, - 2.3424
    .2 + 2.48 + 9.784
-----------------------------------
 +12.4 +48.92,+63.072
    .2 + 2.52
-----------------------------------
 +12.6,+51.44
    .2
-----------------------------------
1+12.8
```

2.3424÷63.272 gives .03; hence 3 is is the next figure of the root, and transforming:

```
1+12.8  +51.44   +63.072    -2.3424    | .03
    .03 +  .3849 + 1.554747 +1.93880241
---------------------------------------------
 12.83 +51.8249 +64.626747 , - .40359759
    .03 +  .3858 + 1.566321
---------------------------------------------
 12.86 +52.2107,+66.193068
    .03 +  .3867
---------------------------------------------
 12.89,+52.5974
    .03
---------------------------------------------
1+12.92
```

.40359759÷66.193068 gives .006, and 6 is the next figure of the root. Again transforming:

```
1+12.92  +52.5974  +66.193068   - .40359759    | .006
    .006 +  .077556 +  .316049736 +.399054706416
----------------------------------------------------
 12.926 +52.674956 +66.509117736,—.004542883584
    .006 +  .077592 +  .316515288
----------------------------------------------------
 12.926 +52.752548 +66.825633024
    .006 +  .077628
----------------------------------------------------
 12.932,+52·830176
    .006
----------------------------------------------------
1+12.938
```

and .004542883584÷66.825633 gives .00006; hence 06 are the next two figures of the root. Again transforming:

```
1+12.944  +52.830176    +66.825633          —.004542883656        | .00006
   .00006+  .0007766436  +  .0031698571586I6 +.0039097281714295I696
-------------------------------------------------------------------------
 12.94406+52.8309526436  +66.828802857158616 ,—.000633155484570483O4
   .00006+  .0007766472  +  .003169903758499 2
-------------------------------------------------------------------------
 12.94412+52.83172930832,+66.83197276091715 2
```

As we will not carry the approximation nearer than to 6 figures, it will not be necessary to go further than has been done, since we have now the means of getting the sixth figure, which is done by dividing .0005331154845704304 by 66.8319727609171152, and we get a number between 7 and 8, but being nearer to the latter, we put 8 as the sixth figure of the root, and we have 5.236068. In the same manner we can find the other roots.

**Art. 103.** But as the number of decimal places becomes inconveniently large, especially when a considerable number of decimal places are desired in the root itself, we must attempt some measure of relief. This may be had by simply using no more places of decimals than are necessary in each stage of the operations.

Having decided on the number of decimal places that shall be in the root, we will remember that that number, or one or two more, will be sufficient to have in the dividends. Also that the number in the dividend minus the number of the place of the required figure of the root at any stage, will give the number of places that ought to be used in the divisor. Thus, if there are to be 6 places of decimals in the approximate root and we are multiplying by the third figure, if the other factor, which is the divisor, has 3 places, the product will contain 6 places and give the dividend to the necessary extent. One or two places more may well be preserved, and all the others to the right dropped; but in multiplications of such reduced numbers we must at the first product on the right hand in every case, add on the figure which would have been "carried" there had no figures been dropped.

If the number of places of decimals can be thus curtailed in the divisor, and since that divisor is itself a product in which the last figure of the root is a factor, the coefficient preceding may be cut down to a still smaller number of places. Each coefficient, as we proceed from right to left, may have one figure more dropped than was done in the case of its immediate predecessor. In this way the coeffi-

cients in the left hand columns will soon and successively become constant, because all decimals would have to be rejected, until finally there may be left only the absolute term and also the penultimate coefficient, which latter simply loses one figure from the right every time a new figure in the root is found.

**Art. 104.** This matter may be illustrated by finding again the root 5.236068:

| | | | | | |
|---|---|---|---|---|---|
| 1 | —8 | +14 | +4 | —8 | 5.236068 |
| | +5 | —15 | —5 | —5 | |
| | —3 | — 1 | —1 | ,—13* | |
| | +5 | +10 | +45 | 10.6576 | |
| | +2 | + 9 | ,+44* | —2.3424* | |
| | +5 | +35 | 9.288 | 1.9388024 | |
| | +7 | ,+44* | 53.288 | — .4035975* | |
| | +5 | 2.44 | 9.784 | .3990549 | |
| 1* | +12* | 46.44 | 63.072* | —.0045426* | |
| | 0.2 | 2.48 | 1.554747 | .0040095 | |
| | 12.2 | 48.92 | 64.626747 | .0005331 | |
| | 0.2 | 2.52 | 1.566321 | | |
| | 12.4 | 51.44* | 66.19306* | | |
| | .2 | .3849 | .31608 | | |
| | 12.6 | 51.8249 | 66.50915 | | |
| | .2 | .3858 | .31656 | | |
| 1* | +12.8* | 52.2107 | 66.826 | | |
| | .03 | .3867 | | | |
| | 12.83 | 52.5974* | | | |
| | .03 | .08 | | | |
| | 12.86 | 52.68 | | | |
| | .03 | .08 | | | |
| | 12.89 | 52.76 | | | |
| | .03 | | | | |
| 1 | +12.92* | | | | |
| | .006 | | | | |
| | 12.926 | | | | |

The places in which decimal figures have been dropped off, and partial amends made by increasing the last figure, will be perceived upon inspection.

Next let the root of which the first figure is the whole numder 2 be found.

| | | | | |
|---|---|---|---|---|
| 1—8 | +14 | +4 | — 8 | 2.7320508 |
| +2 | —12 | 4 | 16 | |
| —6 | 2 | 8 | + 8 | |
| 2 | —8 | —12 | — 7.4599 | |
| —4 | —6 | — 4 | .5401 | |
| 2 | —4 | — 6.657 | —.50511759 | |
| —2 | —10 | —10.657 | .03498241 | |
| 2 | .49 | — 5.971 | —.03411504 | |
| 0 | —9.51 | —16.628 | .00086737 | |
| 0.7 | .98 | — .209253 | —.00085356 | |
| 0.7 | —8.53 | —16.837253 | .00001381 | |
| .7 | 1.47 | — .206679 | .0001366 | |
| 1.4 | —7.06 | —17.04393 | .0000015 | |
| .7 | — .0849 | — .01359 | | |
| 2.1 | —6.9751 | —17.0575 | | |
| .7 | .0858 | — .0135 | | |
| 2.8 | —6.8893 | —17.0711 | | |
| .03 | .0867 | | | |
| 2.83 | —6.802 | | | |
| .03 | .008 | | | |
| 2.86 | —6.794 | | | |
| .03 | | | | |
| 2.89 | | | | |
| .03 | | | | |
| 2.92 | | | | |

The quotient of 8 divided by —4 gives —2, which is much too small, as may be found by trial; it must be increased, and it is found that 2.7 and 2.8, when substituted for $x$ in the first member, give different signs, hence there is a root between them, and we take .7 for the second figure of the root. We afterwards proceed as usual. .7 is a quantity too great to have its second and higher powers dropped as inappreciable.

The root of which .7 is the first figure may be found thus:

| | | | | |
|---|---|---|---|---|
| 1—8 | +14 | +4 | —8 | .763932 |
| .7 | —5.11 | 6.223 | 7.1561 | |
| —7.3 | 8.89 | 10.223 | —.8439* | |
| .7 | —4.62 | 2.989 | .79211376 | |
| —6.6 | 4.27 | 13.212* | —.05178624* | |
| .7 | —4.13 | — .010104 | .03951341 | |
| —5.9 | .14* | 13.201896 | —.01227283* | |
| .7 | —.3084 | — .028392 | .1184220 | |
| —5.2 | —.1684 | 13.173504* | —.0043063* | |
| .06 | —.3048 | — .002368 | .000039472 | |
| —5.14 | —.4732 | 13.171136 | .0042668* | |
| .06 | —.3012 | — 2412 | | |
| —5.08 | —.7744* | 13.15872 | | |
| ·06 | —.0149 | — 72 | | |
| —5.02 | —.789 | 13.158 | | |
| .06 | — 15 | -- 07 | | |
| 1—4.96* | —.804 | 13.15,7,3 | | |

To obtain the fourth root, which is negative, we must transform the equation into another whose negative roots

correspond to the positive roots of this, and conversely, by changing the alternate signs, and we have as follows:

| | | | | |
|---|---|---|---|---|
| 1+8 | +14 | —4 | —8 | .7320508 |
| 0.7 | 6.09 | 14.063 | 7.0441 | |
| 8.7 | 20.09 | 10.063 | — .9559, | |
| .7 | 6.58 | 18.669 | — .89261841 | |
| 9.4 | 26.67 | 28.732, | — .06328159, | |
| .7 | 7.07 | 1.021947 | .06171029 | |
| 10.1 | 33.74, | 29.753947 | — .00157130 | |
| .7 | .3249 | 1.031721 | 154632 | |
| 10.8. | 34.0649 | 30.785668, | — .00002498 | |
| .03 | .3258 | .069478 | 2473 | |
| 10.83 | 34.3907 | 30.85514,6, | 25 | |
| .03 | .3267 | .06952 | | |
| 10.86 | 34 7174, | 30.92467, | | |
| .03 | .0218 | 173 | | |
| 10.89 | 34.739,2 | 30.926,40 | | |
| .03 | .022 | 2 | | |
| 10.92 | 34.75,1 | 30.9,2,8 | | |

The algebraic sum of these roots is equal to 8, the coefficient of the second term with its sign changed, as should be the case:

| |
|---|
| 5.236068 |
| 2.732050 8 |
| .763932 |
| 8.732050 8 |
| —.732050 8 |
| 8. |

EXAMPLES.

1. Find one root of $x^3-2x-5=0$. Ans. 2.0945515.
2. Of $x^3+10x^2-24x-240=0$. Ans. 4.898979.
3. Of $x^5+2x^4+3x^3+3x^2+5x-54321=0$. Ans. 8.414455.

NEWTON'S METHOD.

**Art. 105.** In this mode of approximating to the roots of numerical equations, it will be assumed that all the roots which are equal or which are whole numbers, have been found and divided out. Then having in some way, by Sturm's Theorem, by chance or otherwise, found a a number, or numbers, which differ but slightly from the roots sought, let this approximation, plus or minus a new unknown quantity be substituted for the original unknown quantity in the first member of the equation, and let all the indicated operations be performed. The new unknown quantity represents the difference between the approximate root which is being tried and the true root, and of course should be so small a fraction that all terms involving its powers higher than the first may be dropped as inappreciable, or at all events, producing no serious error.

Then from this equation of the first degree, find the value of this difference, and add it to the trial root when that is too small, or subtract it when the trial root requires to be diminished. We have now an approximate root once corrected. But if this be not close enough to the truth, *let it be used as a trial root*, precisely as before, and the value of a second correction obtained.

And this corrected root may be used for a third correction, and so on to any desired extent.

Let us take as an example

$$x^3+6x^2+x-10=0;$$

in which it has been found by trial that 1.1 is an approximate root, being somewhat too small. Let $u=$ the difference between 1.1 and the true root; we will then have $x=u+1.1$. For a moment let 1.1 be represented by $r$ $\therefore$

$$x=r+u;\ x^2=r^2+2ru+u^2;\ x^3=r^3+3r^2u+3ru^2+u^3;$$

and we have:

$$\left.\begin{array}{rl} x^3 = & u^3+3u^2r+3ur^2+r^3 \\ +6x^2 = & \phantom{u^3+}6u^2+12ur+6r^2 \\ +x = & \phantom{u^3+3u^2r+3ur^2+}u+r \\ -10 = & \phantom{u^3+3u^2r+3ur^2+}-10 \end{array}\right\}=0;$$

and dropping the terms involving $u^2$ and $u^3$, we have:

$(3r^2 + 12r + 1)\, u + r^3 + 6r^2 + r - 10 = 0$, whence $u = \dfrac{10 - r - 6r^2 - r^3}{3r^2 + 12r + 1}$ and restoring 1.1 in place of $r$ we have

$u = \dfrac{10 - 1.1 - 7.26 - 1.331}{17.83} = .0173303..$ and $x = r + u = 1.1173303$. Now if we substitute this value of the root for $r$ in the formula above we would get a second value of $u$ or a second correction. But taking 1.1173 as being one root and dividing it out we get

$x^2 + 7.1173x + 8.95215929 = 0$; and solving this

$x = -3.55865 \pm \sqrt{-8.95215929 + (-3.55865)^2}$ $\therefore$

| | | |
|---|---|---|
| the other two roots are | $-5.48526$ | |
| | $-1.63204$ | to which |
| adding the first | $+1.11730$ | |
| | $-6.00000$ | which |

is the coefficient of the 2d term with its sign changed as it ought to be. The product of these roots which ought to be 10 is 10.002252 + .... showing a good approximation.

### EXAMPLE 2.

Find a root of $x^3 + x^2 + x - 100 = 0$. An approximate root is 4.2. The first correction by Newton's Method gives 4.265.. which is a little too large and a second correction gives 4.264430. The same root found by Horner's Method is 4.2644299..

### FOURIER'S CONDITIONS.

**Art. 106.** It may happen that after several corrections have been made by Newton's Method it will be found that we had approached the true value of the root for a while and then receded from it; this arises from over correction. The value of $u$, the correction, was obtained approximately, as will be remembered, by dropping its higher powers, and

it may have thus happened that the value of $u$ in some case was too large and when added to the approximate root has carried the value beyond the truth; and if this be again corrected in the same sense the new result will be still further off; and would be an approach to some other root.

It is necessary then to know, not only that there is a root between certain limits, but also that there is *no other root* within those same limits. In fact it has been demonstrated by Fourier that

1. The limits between which the required root exists must be so narrow as to contain no other root of the given equation; nor yet of the other two equations obtained by putting the first and second derived polynomials equal to zero.

2. That the approximation must start *from that value which makes the first member and its second derived polynomial have the same sign.*

But it is not deemed necessary to give the demonstration of these principles because their application takes away from that simplicity and expeditiousness which are characteristic of the method of Newton. By that method, as it is given above and as it was left by its immortal author, good, practical results can always be obtained. If at any time it is suspected that the approximate root is departing from the true value instead of approaching it, the matter may be determined at once by substituting in the proposed equation.

If the student is desirous of taking more trouble than is involved in the simple application of Newton's Method it would probably be better at once to apply Horner's Method.

The demonstration of Fourier's Conditions may be found in Hackley's Algebra, Todhunter's Theory of Equations and elsewhere.

# CHAPTER IX.

## Trigonometrical Solution of Equations of the Third Degree.

**Art. 107.** The *Irreducible Case* of these equations may be solved by calling in the aid of Trigonometry. From that branch of mathematics we know that, calling $u$ any arc,

$$\cos 2u = 2\cos^2 u - 1, \ldots\ldots (1)$$

Now, $\cos 3u = \cos(u+2u) = \cos u \cos 2u - \sin u \sin 2u$, and this is equal to $\cos u \cos 2u - \sin u\, 2\sin u \cos u = \cos u(\cos 2u - 2\sin^2 u)$. But $2\sin^2 u = 1 - \cos 2u$, and by substitution in the last expression, we have:

$$\cos 3u = 2\cos 2u \cos u - \cos u, \ldots\ldots (2)$$

Substitute in (2) the value of $\cos 2u$ from (1), and $\cos 3u = 4\cos^3 u - 3\cos u$, whence we derive:

$$\cos^3 u - \tfrac{3}{4}\cos u - \tfrac{1}{4}\cos 3u = 0, \ldots\ldots (3)$$

Now suppose that we had a cubic equation to solve which gave rise to the irreducible case. It may be placed in the form

$$x^3 + px + q = 0, \ldots\ldots (4)$$

and since its roots are all real, $\dfrac{p^3}{27} > \dfrac{q^2}{4}$, and $p$ is negative.

Let $x = r\cos u$, wherein $r$ is a constant yet to be determined. Then $\cos u = \dfrac{x}{r}$, and substituting in (3) we have $\dfrac{x^3}{r^3} - \dfrac{3x}{4r} - \dfrac{1}{4}\cos 3u = 0$, whence

$$x^3 - \frac{3r^2}{4}x - \frac{r^3}{4}\cos 3u = 0, \ldots\ldots (5)$$

By comparing (4) and (5), we see that $\frac{3r^2}{4} = -p$ $\therefore$ $\frac{r^2}{4} = \frac{-p}{3}$ and $r = 2\sqrt{\frac{-p}{3}}$. Also that $\frac{r^3\cos 3u}{4} = -q$; $\therefore$ $\frac{r^2}{4} \times r\cos 3u = -q$, or $\frac{-p}{3} \times r\cos 3u = -q$; $\therefore$ $\cos 3u = \frac{3q}{pr} = \frac{3q}{p.2\sqrt{\frac{-p}{3}}} = \frac{q}{2\sqrt{\frac{-p^3}{27}}}$. Now, since $q$ and $p$ are known and $p$ is negative, we know the numerical value of the cosine of $3u$ and having found $3u$ from the Tables of Natural Sines and Cosines, we know $u$ and can find its cosine. This being multiplied by $r$ gives the value of $x$, which is a root of the proposed equation. The numerical value of $r$ comes of course from $r = 2\sqrt{\frac{-p}{3}}$.

Thus we have one of the real roots; but the value $\frac{q}{2\sqrt{\frac{-p^3}{27}}} =$ not only $\cos 3u$, but also $\cos(360° - 3u)$ and $\cos(360° + 3u)$, consequently the cosines of the thirds of these latter two arcs will be the remaining roots, after having been multiplied by $r$.

**Art. 108.** Should it happen that $3u = 180°$ or any multiple thereof, $\cos(360° - 3u)$ and $\cos(360° + 3u)$ would be equal, and the roots corresponding would be equal; but they might have been discovered and divided out in the first instance, when no resort to Cardan's Formula would have been necessary.